A Comprehensive Study On Composite Physical Systems And Their Application In Information Protocols

Thesis submitted for the partial fulfillment of the requirements for the degree Doctor of Philosophy in Science

by

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Department of Physics
Faculty of Natural and Mathematical Sciences
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Kolkata, India

2025

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I dedicate this thesis with profound love and gratitude to my parents, Pooja Gopalkrishna Naik and Gopalkrishna Savalo Naik, whose unwavering support and enduring commitment to my education have been my greatest strength. I am deeply indebted to my brother and teacher, Siddhu, whose thoughtful conversations and guidance have profoundly shaped my academic journey. I also dedicate this work to my dear friend Kunika, whose constant encouragement and steadfast presence have been an invaluable source of strength throughout the course of my doctoral studies.

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Declaration

I hereby declare that this thesis contains original research work carried out by me under the guidance of Dr. Manik Banik, Associate Professor, Department of Physics of Complex Systems, S.N. Bose National Centre for Basic Sciences (SNBNCBS), Kolkata, India as part of the PhD programme.

All information in this document have been obtained and presented in accordance with academic rules and ethical conduct.

I also declare that, as required by these rules and conduct, I have fully cited and referenced all materials and results that are not original to this work.

I also declare that, this work has not been submitted for any degree either in part or in full to any other institute or University before.

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CERTIFICATE

This is to certify that the thesis entitled "A Comprehensive Study On Composite Physical Systems And Their Application In Information Protocols" submitted by **Shri Sahil Gopalkrishna Naik**, Registration Number **RC001-23RS209110003** and date of registration 18th January, 2024, in partial fulfilment of the requirements for the award of "Doctor of Philosophy" is a record of bona-fide research work carried out by him under my supervision.

Neither his thesis nor any part of the thesis has been submitted for any degree/diploma or any other academic award anywhere before.

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Abstract

Understanding why nature privileges quantum theory over other conceivable physical frameworks remains one of the most profound challenges in the foundations of physics. In pursuit of this question, we begin by examining how different composition rules between quantum systems influence the correlations that may arise in experimental settings. While previous studies have established that no bipartite composition yields spacelike correlations exceeding those allowed by quantum theory, we demonstrate that, in the domain of timelike correlations, certain composition rules can generate correlations stronger than those permitted within quantum theory. We further analyze the implications of such correlations for a communication task called the *Pairwise distinguishability game*, thereby providing a potential route toward experimental testing of the actual composition rule realized in nature.

Next, we turn to the composition of non-quantum systems, which provides a platform for contrasting quantum and non-quantum correlations and, in turn, for gaining deeper insights into the nature of quantum correlations themselves. We investigate Hardy-type nonlocality within a broad class of operational theories whose local state spaces are regular polygons. We develop a systematic characterization of entangled states in these models and identify those that can exhibit Hardy nonlocality. Our results reveal that, unlike the case of two-qubit systems where mixed entangled states cannot demonstrate Hardy nonlocality, in polygon models mixed entangled states can indeed exhibit such correlations. We further uncover the role of a specific dynamical symmetry—prepare–measure reciprocity—whose absence plays a crucial role in enabling mixed-state Hardy nonlocality.

Composition of physical systems give rise to a wide arena of plausible theories. However more general approaches to composition particularly to spacetime regions can yield exotic phenomenon in quantum theory itself. Considering the composition of spacetime events, the traditional approach yields a spacetime, where every pair of events is either timelike- or spacelike-related. By contrast, employing the framework of higher-order quantum theory introduces the notion of indefinite composition of events. Treating causal indefiniteness as a potential information-theoretic resource, we analyze its role in the Data Retrieval (DR) task, in which parties must collaboratively extract classical information encoded in quantum states. We show that indefinite causal order can enhance performance in the DR task beyond what is possible under definite causal structures. We further establish a formal equivalence with the well-studied Guess Your Neighbor's Input causal game, in terms of the optimal success probability. To characterize useful processes for a particular variant of the DR game, we derive a Peres-like "positive under partial transpose" criterion which provides further classification among the set of process matrices. We also report a super activation phenomenon, whereby two individually useless processes become useful for the DR task when combined. Extending our analysis to the tripartite setting, we show that classical causally nonseparable processes can outperform quantum bi-causal ones.

Finally, we investigate the classical simulation of quantum processes, particularly in the context of quantum channel simulation. Unlike standard formulations, which typically involve a single sender and receiver, we address the efficient classical simulability of quantum channels in network scenarios. In such networks, receivers often need to perform composite measurements on multiple quantum systems supplied by different senders. We prove that perfect classical simulation of a qubit channel is impossible with any finite amount of bidirectional classical communication, even when shared randomness is allowed—thereby exposing a fundamental gap between classical and quantum capabilities. We identify entangled-effect statistics as the source of this gap, since separable measurements in generic network configurations turn out to be efficiently classically simulable. Extending our analysis to noisy channels, such as depolarizing channels, we show that efficient classical simulation is achievable; however, the required communication cost diverges as the noise parameter vanishes.

List of publications

Publications and communicated works relevant to the thesis:

- "*Composition of Multipartite Quantum Systems: Perspective from Timelike Paradigm*": **Sahil Gopalkrishna Naik**, Edwin Peter Lobo, Samrat Sen, Ram Krishna Patra, Mir Alimuddin, Tamal Guha, Some Sankar Bhattacharya, and Manik Banik; Phys. Rev. Lett. **128**, 140401,(2022).

- "*Bipartite polygon models: entanglement classes and their nonlocal behaviour*": Mayalakshmi K, Thigazholi Muruganandan, **Sahil Gopalkrishna Naik**, Tamal Guha, Manik Banik, Sutapa Saha; Quantum 9, 1599 (2025).

- "*Harnessing Causal Indefiniteness for Accessing Locally Inaccessible Data*": **Sahil Gopalkrishna Naik**, Samrat Sen, Ram Krishna Patra, Ananya Chakraborty, Mir Alimuddin, Manik Banik, and Pratik Ghosal; (Communicated in New Journal of Physics) arXiv:2407.20543 [quant-ph]

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- "*Distilling nonlocality in quantum correlations*": **Sahil Gopalkrishna Naik**, Govind Lal Sidhardh, Samrat Sen, Arup Roy, Ashutosh Rai, and Manik Banik; Phys. Rev. Lett. **130**, 220201,(2023).

- "*Quantum Incompatibility in Parallel vs Antiparallel Spins*": Ram Krishna Patra, Kunika Agarwal, Biswajit Paul, Snehasish Roy Chowdhury, **Sahil Gopalkrishna Naik**, Manik Banik; Accepted in Phys. Rev. Lett. (2026) (arXiv:2504.04723).

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- "*Certifying beyond quantumness of locally quantum no-signaling theories through a quantum-input Bell test*": Edwin Peter Lobo, **Sahil Gopalkrishna Naik**, Samrat Sen, Ram Krishna Patra, Manik Banik, and Mir Alimuddin; Phys. Rev. A **106**, L040201,(2022).

- "*Timelike correlations and quantum tensor product structure*": Samrat Sen, Edwin Peter Lobo, Ram Krishna Patra, **Sahil Gopalkrishna Naik**, Anandamay Das Bhowmik, Mir Alimuddin, and Manik Banik; Phys. Rev. A **106**, 062406 ,(2022).

- "*Advantage of Hardy's Nonlocal Correlation in Reverse Zero-Error Channel Coding*": Mir Alimuddin, Ananya Chakraborty, Govind Lal Sidhardh, Ram Krishna Patra, Samrat Sen, Snehasish Roy Chowdhury, **Sahil Gopalkrishna Naik**, Manik Banik; Phys. Rev. A **108**, 052430 (2023).

- "*Classical analogue of quantum superdense coding and communication advantage of a single quantum system*": Ram Krishna Patra, **Sahil Gopalkrishna Naik**

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- "*When Mei-Gu Guan's 1960 postmen get empowered with Bell's 1964 nonlocal correlations: Nonlocal advantage in vehicle routing problem*": Abhishek Banerjee , Pratapaditya Bej, Amit Mukherjee , **Sahil Gopalkrishna Naik** , Mir Alimuddin , Manik Banik; APL Quantum **1, 036105 (2024)**.

- "*Scalable and Noise-Robust Communication Advantage of Multipartite Quantum Entanglement*": Ananya Chakraborty, Ram Krishna Patra, Kunika Agarwal, Samrat Sen, Pratik Ghosal, **Sahil Gopalkrishna Naik**, Manik Banik; Phys. Rev. A **111, 032617(2025)**.

- "*Unlocking the advantage of qubit communication in multi-node network configurations*": Ananya Chakraborty, **Sahil Gopalkrishna Naik**, Edwin Peter Lobo, Ram Krishna Patra, Samrat Sen, Mir Alimuddin, Amit Mukherjee and Manik Banik; New J. Phys. **27** 023027 (2025).

- "*Advantage of Greenberger-Horne-Zeilinger correlation in communication complexity: A step beyond the conventional Greenberger-Horne-Zeilinger paradox*": Ananya Chakraborty, Kunika Agarwal, **Sahil Gopalkrishna Naik**, Manik Banik; Phys. Rev. A **112**, 012214 (2025).

- "*Detecting genuine multipartite entanglement using moments of positive maps*": Saheli Mukherjee, Bivas Mallick, **Sahil Gopalkrishna Naik**, Ananda G. Maity, Archan S. Majumdar; Phys. Rev. A **112**, 062428 (2025).

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- "*Nonlocality-Assisted Enhancement of Error-Free Communication in Noisy Classical Channels*": Kunika Agarwal, **Sahil Gopalkrishna Naik**, Ananya Chakraborty, Samrat Sen, Pratik Ghosal, Biswajit Paul, Manik Banik, Ram Krishna Patra; arXiv:2412.04779v1 [quant-ph].

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Chapter 1

General Introduction

1.1 Motivation

In the scientific method, we observe natural phenomena and attempt to formulate physical theories that accurately describe and predict these observations. One of the earliest and most influential frameworks in this regard was Newtonian mechanics, developed by Isaac Newton to explain the motion of everyday objects. His formulation of the law of universal gravitation emerged similarly—from empirical observations and the effort to encapsulate them within a consistent theoretical structure.

Later, Einstein's theories of special and general relativity addressed inconsistencies that arose when applying Newtonian mechanics to high-speed motion and strong gravitational fields. These classical theories provided a remarkably coherent and predictive framework for understanding macroscopic phenomena in the natural world.

However, in the early 20th century, several experimental results revealed limitations of classical mechanics, particularly at the microscopic scale. Experiments such as the double-slit interference experiment, the Stern–Gerlach experiment, and the photoelectric effect, among others, suggested that classical physics was insufficient to explain certain fundamental features of nature. These anomalies prompted the development of quantum mechanics—a profoundly successful and predictive theory that remains one of the cornerstones of modern physics.

This shift underscores a broader theme: the overarching goal of scientific inquiry is to systematically expand our understanding of the universe, often by developing models that explain previously unaccounted-for phenomena.

To illustrate the process of modeling physical systems, consider a hypothetical system, denoted as Sys-1, which is understood to some extent. By a physical system, we refer to any portion of the universe isolated for study. To say that a system is "understood to some extent" means that we can predict some of its measurable properties, either deterministically or probabilistically. This understanding, however, may be incomplete; certain behaviors exhibited by the system may still need further explanation.

Despite such incompleteness, scientific modeling proceeds by assuming the existence of such systems and formalizing our partial understanding of them. Now suppose we introduce a second physical system, Sys-2, which similarly might be only partially understood. A foundational question that arises in this context is:

> *Given our current understanding of the individual systems Sys-1 and Sys-2, what can be inferred about our understanding of the composite system Sys-1 ∪ Sys-2?*

Several key sub-questions emerge from this line of inquiry:

- If our understanding of both subsystems were complete, would it necessarily imply complete understanding of the composite system?
- Is the composite system Sys-1 ∪ Sys-2 a well-defined physical system in its own right? (See Fig.(1.1))
- If so, is there a unique way to define the composition of two systems?
- If multiple composition rules are mathematically valid, are all of them physically meaningful, or only a restricted subset?

These questions lie at the heart of foundational investigations in physics, particularly within the frameworks of quantum mechanics, generalized probabilistic theories (GPTs), and the study of complex and emergent systems. They probe the structural underpinnings of physical theories and examine the extent to which these theories can consistently describe not only isolated subsystems, but also their interactions and joint behavior.

The central objective of this thesis is to demonstrate that the study of composite systems—particularly the nature of their composition—can yield profound

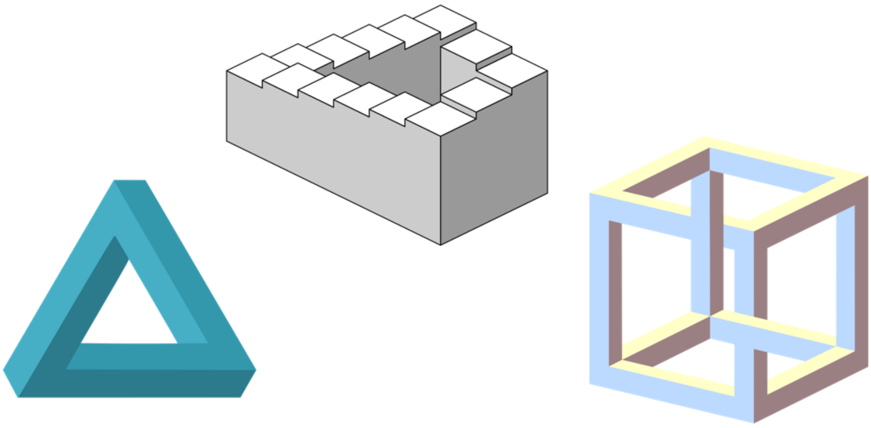

FIG. 1.1 (Left) Penrose tribar: An impossible figure formed by the illusory junction of three bars. (Middle) Penrose stairs: A staircase where each step appears to rise above the previous one, yet the path loops endlessly. (Right) Escher's impossible cube: A cube whose edges are arranged in a geometrically inconsistent manner, making it physically unrealizable.

insights into the underlying structure of physical reality. Beyond merely analyzing the statistical properties of composite systems, this work emphasizes the foundational importance of the rules that govern their composition. The specific manner in which subsystems are composed can significantly constrain the class of physically realizable phenomena. As a consequence, the structure of composition itself imposes nontrivial, and often subtle, information-theoretic constraints on the global behavior of the system.

A paradigmatic scenario in the study of composite system statistics arises in the context of spacelike separated experiments. In such scenarios, two experimenters—commonly referred to as Alice and Bob—perform local operations in spacelike separated regions. In 1964, John Bell famously derived the inequality that now bears his name, demonstrating that quantum mechanical predictions for certain correlation experiments are incompatible with any local deterministic theory [1]. That is, no physical theory that adheres to both locality and determinism can reproduce all of the statistical predictions of quantum mechanics.

Experimental confirmation of Bell inequality violations—most notably by the works of Aspect, Clauser, and Zeilinger [2–5]—led to the 2022 Nobel Prize in Physics. Crucially, such violations rely on the presence of entanglement between the quantum systems held by Alice and Bob. This naturally raises a foundational question: What role does the composition rule between quantum systems play in enabling the violation of Bell inequalities?

To formally investigate this question, one can employ the framework of Generalized Probabilistic Theories (GPTs). Within GPTs, composite systems can be constructed using a variety of composition rules. However, under two widely accepted physical principles—(i) no-signaling, i.e., the prohibition of faster-than-light communication, and (ii) local tomography, the requirement that global states can be fully characterized by local measurements—the space of allowable compositions narrows to a continuum bounded by two extremes: the minimal tensor product, which does not allow for entanglement, and the maximal tensor product, which allows for all entangled correlations compatible with no-signaling and local tomography.

Quantum theory occupies a special position within this spectrum: the standard quantum composition rule, defined by the tensor product of Hilbert spaces, is self-dual and lies between the minimal and maximal constructions. A remarkable result by Barnum et al. [6] demonstrates that, despite the diversity of compositional possibilities within GPTs, the set of experimentally accessible correlations from spacelike-separated measurements can always be reproduced using the standard quantum composition rule.

Given that nature exhibits violations of Bell inequalities, one can conclusively exclude composition rules that forbid entanglement (such as the minimal tensor product). However, many other composition rules remain empirically indistinguishable in this domain, leaving open the possibility that nature may employ a non-quantum composition structure consistent with observed nonlocal correlations.

This leads to a profound dilemma: either we accept that the exact composition rule realized in nature may not be empirically accessible from spacelike-separated experiments alone, or we must explore alternative experimental domains. In this context, timelike-separated experiments offer a compelling avenue of investigation.

Consider a scenario in which Alice possesses two quantum subsystems composed according to a specific rule (e.g., minimal, maximal, quantum, or any other). She encodes information into joint states of the composite system and transmits it to another party, who then performs measurements to retrieve the encoded information. A natural extension of the question addressed by Barnum et al. [6] is to ask: Do all composition rules predict equivalent behavior in such timelike operational settings? More specifically, can one distinguish between composition rules based on information-processing tasks involving timelike-separated parties?

These questions motivate the investigations presented in Chapter [3], where we explore whether empirical evidence from timelike experiments can provide stronger constraints on the nature of composition than those available from spacelike correlations alone. Ultimately, this line of inquiry seeks to address whether the standard quantum composition rule—given by the tensor product of Hilbert spaces—can be derived or justified from operational principles observable in temporally ordered experimental settings.

Thus far, our discussion has been restricted to systems composed of quantum subsystems. However, if one relaxes this assumption and allows for non-quantum subsystems, it becomes possible to construct correlations in the spacelike-separated setting that go beyond those permitted by quantum mechanics.

A prominent example of this phenomenon is the Popescu-Rohrlich (PR) box, introduced by Popescu and Rohrlich in [7]. The PR box defines a correlation that is strictly stronger than those allowed by quantum theory, despite respecting the no-signaling principle. Moreover, it is well known that correlations such as those exhibited by the PR box can trivialize communication complexity [8], suggesting that their existence would have drastic consequences for the structure of information processing in nature. These observations raise fundamental questions about the nature of locality, composition, and the operational constraints that limit physically realizable correlations.

It is therefore natural to ask: What constraints can be placed on subsystems by studying the space of allowed spacelike correlations? In Chapter [4], we undertake a systematic investigation of such generalized nonlocal correlations and analyze how various assumptions about local systems and their composition influence the boundary between quantum and post-quantum theories. This analysis provides foundational insight into the structure of quantum correlations and the extent to which they are determined—or constrained—by more primitive physical principles.

The aforementioned studies provide significant insights into the roles of subsystems and their compositions in relation to physically plausible phenomena. However, these analyses typically assume systems in the conventional sense, where subsystems are understood as distinct entities embedded within a fixed spacetime structure. One may, however, extend this perspective and consider spacetime itself as a form of composition—specifically, a composition of events (or subsystems) embedded within a larger spacetime manifold.

In classical and relativistic theories, such a composition is encoded via definite causal structure. That is, for any pair of events A and B, one assumes that either A lies in the causal past of B, B lies in the causal past of A, or the two events are spacelike separated. All classical and relativistic frameworks describe physical phenomena against this fixed causal background, assuming that the spacetime composition of events is well-defined and globally consistent.

However, a central line of inquiry in approaches to quantum gravity questions this assumption. In particular, classical spacetime lacks the capacity to incorporate superpositions of causal structures, a phenomenon naturally suggested by the principle of superposition in quantum mechanics. Lucien Hardy, in [9, 10], was among the first to formalize this idea, proposing a framework that accommodates superposed spacetimes.

Building on this idea, Oreshkov, Costa, and Brukner introduced the framework of process matrices in [11], which generalizes standard quantum theory to allow for indefinite causal order. In this framework, causal relationships between events are not necessarily fixed and can exist in quantum superposition. They also derived causal inequalities, which serve as constraints that any theory with definite causal structure must satisfy. Violations of these inequalities—possible within the process matrix formalism—indicate the presence of genuinely non-classical causal relations.

A particularly striking example is the quantum switch, introduced by Chiribella et al. in [12]. A quantum switch consists of a control qubit in a superposition state, effectively determining the order of two quantum operations in a coherent manner. This is a concrete demonstration of indefinite causal order and shows that quantum theory, when extended appropriately, admits such non-classical compositions of operations and events.

Given the growing interest in indefinite spacetime composition and its potential role in a unified theory of quantum gravity, a natural question arises: What novel information-theoretic tasks become possible in such frameworks—tasks that are impossible under fixed causal structures?

In Chapter [5], we address this question by introducing an information-theoretic task termed the *Data Retrieval Task*. We show that when multiple spacetime regions are composed in an indefinite manner, this composition can provide a fundamental advantage in accessing information that would otherwise remain inaccessible under conventional spacetime structures. This result further underscores the deep interplay between causal structure and information processing in the search for a more complete theory of nature.

When analyzing statistical properties of composite quantum systems, one naturally encounters the notion of entanglement at the measurement level. A paradigmatic example arises in quantum information protocols such as quantum teleportation and superdense coding, both of which rely on a measurement in the Bell basis—an orthonormal basis composed entirely of maximally entangled states. These protocols demonstrate the operational significance of entangled measurements in harnessing the full potential of quantum resources.

In a notable result, Bennett *et al.* [13] constructed a composite system measurement that, while consisting solely of product (i.e., non-entangled) eigenstates, cannot be implemented by performing local measurements on each subsystem individually. This illustrates that even in the absence of entanglement at the level of measurement eigenstates, the measurement itself may still exhibit nonlocal behavior in terms of implementation. This raises a foundational question: What additional resources, when supplemented with local operations, are necessary to simulate such measurements?

Measurements of this kind often emerge in quantum network scenarios, where multiple independent senders transmit quantum states to a common receiver. The receiver, in turn, may wish to extract global (i.e., composite) information by performing a joint measurement across all received systems. Given the practical constraints and theoretical interest, it becomes pertinent to ask: Can certain quantum channels in such networks be substituted with classical channels without loss of functionality? Moreover, what are the limitations if the receiver is restricted to performing only trivial composite measurements, i.e., measurements that are effectively local in nature?

These questions motivate the study undertaken in Chapter [6], where we analyze the simplest nontrivial network configuration comprising two independent senders and a single receiver. Our goal is to investigate the extent to which one of the quantum channels connecting a sender to the receiver can be simulated using classical communication, while retaining a quantum channel between the other sender and the receiver. This minimal scenario serves as a foundational model to explore the interplay between classical simulation of quantum channels and composite measurements.

1.2 Brief Outline of the Thesis:

- **Chapter [2]** introduces the essential preliminaries and mathematical tools that are extensively used throughout the subsequent chapters.
- **Chapter [3]**, based on the work in [14], investigates the role of composition in time-like correlation experiments, analyzing how composition of quantum systems in the context of general probabilistic theories can lead to post quantum correlations depending on the composition.
- **Chapter [4]**, based on [15], explores the nonlocal features of composite systems within composite polygon systems studied widely in the framework of Generalized Probabilistic Theories (GPTs). The chapter discusses the foundational implications of the spacelike correlations that arise from such composite structures.
- **Chapter [5]**, based on [16], presents an information-theoretic task designed to contrast causal and noncausal processes. It is shown that certain processes lacking any causal interpretation outperform those constrained by definite causal order.
- **Chapter [6]**, based on [17], addresses the question of whether a quantum channel can be classically simulated in general network scenarios. A no-go theorem is established, demonstrating that such a classical substitution is fundamentally impossible in general cases.
- **Chapter [7]** summarizes the main findings of the thesis and outlines possible directions for future research.

Chapter 2

Preliminaries

2.1 Quantum Theory

Until the early 20th century, classical physics—embodied by Newtonian mechanics and Maxwell's equations—provided a remarkably successful framework for explaining a wide range of physical phenomena. However, a growing body of experimental evidence began to reveal fundamental discrepancies that classical theories could not resolve, ultimately leading to the birth of quantum mechanics. The journey began with Planck's study of blackbody radiation in 1900, which introduced the revolutionary concept of energy quantization. This was soon followed by Einstein's explanation of the photoelectric effect in 1905, which demonstrated the particle-like behavior of light and further challenged the classical wave theory of electromagnetism.

Subsequent experiments continued to expose the inadequacy of classical physics. Bohr's atomic model (1913) and the Franck-Hertz experiment (1914) provided compelling evidence for discrete atomic energy levels. Compton scattering (1923) confirmed the momentum transfer associated with photons, while the Stern-Gerlach experiment (1922) revealed the quantization of angular momentum, or spin. Wave-particle duality—a central feature of quantum mechanics—was experimentally verified through the Davisson-Germer electron diffraction experiment and the electron double-slit experiment, both conducted in 1927.

The philosophical implications of quantum theory were brought to the forefront by the EPR paradox (1935), which questioned the completeness of quantum mechanics. Bell's theorem (1964), and its experimental validation by Aspect and collaborators in the 1980s, established the phenomenon of quantum entanglement, challenging classical notions of locality and realism.

Concurrently, the theoretical framework of quantum mechanics was being rigorously developed. De Broglie's hypothesis (1924) extended wave-particle duality to matter, Heisenberg (1927) introduced the uncertainty principle, and Schrödinger (1926) formulated wave mechanics. Dirac's work (1928) unified quantum mechanics with special relativity and predicted the existence of antimatter, while Feynman's path integral formalism in the 1940s revolutionized quantum electrodynamics and provided new computational tools.

To place quantum theory on a firm mathematical footing, von Neumann and Hilbert introduced an axiomatic framework that defined quantum states, observables, and their evolution within the structure of Hilbert space. Their formulation established a consistent and rigorous foundation for quantum mechanics, which we now proceed to define formally in the following subsection.

2.1.1 Postulates of Quantum Mechanics

- **Postulate I:** *Associated to any isolated physical system S is a complex vector space with inner product (that is, a Hilbert space). The Hilbert space denoted by H is always taken to be separable i.e. of countable dimension. All properties of the system are completely described by a density operator acting on the associated Hilbert space.*

  Definition 1. ***Positive semi definite operator:*** *A linear operator $P : H \to H$ is called a positive semi definite operator if any of the following equivalent conditions hold:*

  - $\langle\psi|P|\psi\rangle \geq 0 \ \forall \ |\psi\rangle \in H$.
  - $P = P^{\dagger}$ *and all eigenvalues of P are nonnegative.*

  If we have $\langle\psi|P|\psi\rangle > 0 \ \forall \ |\psi\rangle$ then the linear operator is termed positive definite. We refer to positive semi definite operators simply as positive operators. If needed the definiteness condition is explicitly mentioned. For notational convenience we denote a positive semi definite operator or positive definite operator by writing $P \geq 0$ or $P > 0$ respectively.

  Definition 2. ***Density operator:*** *A positive operator ρ is termed as a density operator if* $\mathrm{Tr}[\rho] = 1$.

  Although elementary texts in quantum mechanics state that the state is completely described by a unit vector $|\psi\rangle$ in the Hilbert space H, this notion of describing states does not capture the full description of the state as it is indeed completely reasonable to ask what is the the associated state for a system which is prepared in a mixture of two separate states $|\psi\rangle$ and $|\psi'\rangle$. To deal with such cases one shifts to the notion of density operators. This gives rise to the concept of pure and mixed states.

Definition 3. ***Pure state:*** *A density operator ρ is said to be a pure state if it cannot be written as a convex mixture of other density operators. Mathematically $\rho = \sum_i p_i \rho_i$ implies $\rho_i = \rho \ \forall \ i$. Here $\{p_i\}_i$ denotes a probability vector. Equivalently we can also write $\rho = |\psi\rangle \langle\psi|$ for some unit vector $|\psi\rangle$ or $\rho^2 = \rho$ or $\mathrm{Tr}[\rho^2] = 1$.*

A density operator that is not pure is termed as a mixed state. In such cases we can decompose this density operator as convex mixture of other pure states. For mixed states we have $\rho - \rho^2 \geq 0$ or $\mathrm{Tr}[\rho^2] < 1$.

- **Postulate II:** The evolution of a closed quantum system is described by a unitary transformation $\mathscr{U}$.The action of $\mathscr{U}$ on a arbitrary state ρ is given by: $\mathscr{U}(\rho) = U\rho U^\dagger$,where U is a unitary operator acting on H. That is, the state ρ of the system at time t_1 is related to the state ρ' of the system at time t_2 as $\rho' = U\rho U^\dagger$ where U depends only on the times t_1 and t_2.

  Postulate II states that a system must be closed, with no external interactions. While all real systems interact with the environment to some extent, many can be approximated as closed and follow unitary evolution. Any open system can also be seen as part of a larger, closed, unitarily evolving system. Later, we will introduce tools to describe open system dynamics.

  Closed quantum systems evolve unitarily, but interactions with measurement devices make the system open, breaking this unitary evolution. Postulate 3 explains how measurements affect quantum systems.

- **Postulate III:** Quantum measurements are described by a collection $\{M_m\}$ of measurement operators acting on H. The index m refers to the measurement outcomes that may occur in the experiment. If the state of the quantum system is ρ immediately before the measurement then the probability that result m occurs is given by the Born's rule

$$p(m) = \mathrm{Tr}[M_m \rho M_m^\dagger] \tag{2.1}$$

and the updated state after the measurement is given by

$$\rho_m = \frac{M_m \rho M_m^\dagger}{p(m)} \tag{2.2}$$

Since $\sum_m p(m) = 1 \; \forall \; \rho$ we must have the following completeness relation for $\{M_m\}$

$$\sum_m M_m^\dagger M_m = \mathbb{I} \tag{2.3}$$

where $\mathbb{I}$ denotes the identity operator acting on H.

Projective Measurements: An interesting special case for measurements which are widely used are known as Projective Measurements. If all measurement operators $\{M_m\}$ project onto orthogonal subspaces, then such a measurement is termed as a projective measurement. Mathematically, we have $M_m M_n = \delta_{mn} M_n \; \forall \; m,n$.

POVM Measurements: Postulate 3 of quantum mechanics covers two aspects: the probabilities of different measurement outcomes and the post-measurement state of the system. In many cases, such as single-measurement experiments, only the outcome probabilities matter. For these situations, the POVM (Positive Operator-Valued Measure) formalism provides an effective mathematical tool. Although derived from Postulate 3, POVMs are elegant and widely used, which warrants separate discussion.

If we define $E_m = M_m^\dagger M_m$, then the outcome statics can be represented just by using the positive operators E_m, such as $p(m) = \mathrm{Tr}[M_m \rho M_m^\dagger] = \mathrm{Tr}[E_m \rho]$ (from the cyclic nature of the trace). However, note that the post-measurement state cannot be uniquely defined from E_m as given a particular E_m for a fixed m we might have multiple measurement operators $M_m, M'_m, M''_m, \cdots$ such that we have $E_m = M_m^\dagger M_m = M_m'^\dagger M'_m = M_m''^\dagger M''_m \cdots$. Also, it is evident that any such operator must be positive as: $\langle\psi| E_m |\psi\rangle = \langle\psi| M_m^\dagger M_m |\psi\rangle \geq 0 \; \forall \; m, |\psi\rangle$. Keeping this in mind, a formal definition for POVM Measurements can be given as follows.

Definition 4. ***POVM Measurements:*** *A collection of operators $\{E_m\}$ is a POVM measurement if $E_m \geq 0 \; \forall \; m$ and we have the completeness relation $\sum_m E_m = \mathbb{I}$.*

For a composite quantum system consisting of two or more distinct physical systems, we need a way to describe its overall state. The following postulate explains how the state space of the composite system is constructed from the state spaces of its individual components.

- **Postulate IV:** Given two systems, S_1 and S_2, with associated Hilbert spaces H_1 and H_2 respectively, the Hilbert space of the composite quantum system,H_{comp}, is defined as the tensor product of the individual Hilbert spaces: $H_{\text{comp}} = H_1 \otimes H_2$.

While some argue that the above postulate lacks full justification, and alternative formulations of quantum theory such as the operator algebra framework relax this assumption, we will adhere to this postulate in this thesis. For most systems, particularly finite-dimensional ones, this formulation is sufficient to accurately describe composite quantum systems.

Nearly all physical phenomena can be described using the four postulates outlined above. We now turn our attention to finite-dimensional quantum systems, focusing specifically on the quantum analogue of a classical bit—the qubit.

2.1.2 Finite dimensional classical and quantum systems

Here we review some basic facts about finite dimensional quantum systems which serve as the quantum analogue of finite dimensional classical system. A ($d < \infty$) dimensional classical system simply corresponds to d perfectly distinguishable states of the system. For instance $d = 2$ can correspond to the state of a coin more formally known in the information theory community as a bit. The only pure states for such a coin are heads and tails. We can also have convex mixtures of such states such as a coin prepared in heads state with probability p and in tails state with probability $1 - p$. Geometrically the state space of such a coin represents a segment $p \in [0,1]$ laying in $\mathbb{R}$. Another example in that of a 6 sided dice where the most general state can be represented as a probability vector $(p_1, p_2, \cdots, p_6)^T$ denoting the probabilities of preparing the dice in the states $\{1,2,\cdots,6\}$ respectively. A dice represents a 6 dimensional classical system with it's state space being isomorphic to a 5 dimensional simplex embedded in $\mathbb{R}^5$. The formal definition for a d dimensional classical system is given as

Definition 5. ***d Dimensional classical system:*** *A d dimensional classical system has d perfectly distinguishable pure states denoted by $\{s_i\}_{i=0}^{d-1}$. Any mixed* *state can be expressed uniquely as a convex mixture* $\sum_{i=0}^{d-1} p_i s_i$ *or simply as a probability vector* $(p_0, p_1, \cdots, p_{d-1})^T$*. The state space of such a system is isomorphic to a* $d-1$ *simplex embedded in* $\mathbb{R}^{d-1}$.

The quantum analogue of a d dimensional classical system is a system whose associated Hilbert space is d dimensional. We represent this complex d dimensional Hilbert space as $\mathbb{C}^d$. As said above a d dimensional classical system consists only of d perfectly distinguishable pure states, so at first it might feel unnatural to associate it with a quantum system $\mathbb{C}^d$ consisting of uncountably many pure states $|\psi\rangle$. The reason for this is that although the system associated to $\mathbb{C}^d$ has uncountable pure states it only consists of d perfectly distinguishable states (see also subsection [2.1.5]). We now move to one of the most profound results in Quantum theory:

Distinguishability of quantum states

Theorem 1. *Non orthogonal quantum states cannot be distinguished perfectly. More precisely,* $|\psi_1\rangle, |\psi_2\rangle \in \mathbb{C}^d$ *are perfectly distinguishable iff* $\langle\psi_1|\psi_2\rangle = 0$.

Proof. To distinguish $|\psi_1\rangle$ and $|\psi_2\rangle$ we have to perform a 2 outcome measurement on the given state. If we observe the first outcome we infer that the given state is $|\psi_1\rangle$ and similarly if we observe the second outcome we infer that the given state is $|\psi_2\rangle$. To distinguish the states perfectly we require that the probability for getting the outcome i given the state is $|\psi_1\rangle$ to be $p(i||\psi_j\rangle) = \delta_{ij}$. Where $p(i||\psi_j\rangle)$ denotes conditional probability. As we are only worried about the statistics of the outcomes we can use the POVM formalism and we assume we have two POVM elements $\{\pi_1, \pi_2\}$ with $\pi_1 + \pi_2 = \mathbb{I}$ such that $p(m||\psi_j\rangle) = \langle\psi_j|\pi_i|\psi_j\rangle = \delta_{ij}$. Since $\langle\psi_1|\pi_2|\psi_1\rangle = 0$ we must have $\pi_1 = |\psi_1\rangle\langle\psi_1| + P_1$, with $P_1 \geq 0$ as the $\mathbb{I}$ contains the projector $|\psi_1\rangle\langle\psi_1|$ but π_2 doesn't. Similarly we must also have $\pi_2 = |\psi_2\rangle\langle\psi_2| + P_2$, with $P_2 \geq 0$. Considering the completeness relation we get:

$$
\begin{aligned}
|\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2| + P_1 + P_2 &= \mathbb{I} \\
\langle\psi_1|(|\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2| + P_1 + P_2)|\psi_1\rangle &= \langle\psi_1|\mathbb{I}|\psi_1\rangle \\
1 + |\langle\psi_1|\psi_2\rangle|^2 + \langle\psi_1|(P_1 + P_2)|\psi_1\rangle &= 1
\end{aligned}
\tag{2.4}
$$

The above equation can be satisfied iff $\langle\psi_1|\psi_2\rangle = 0$ as both quantities $|\langle\psi_1|\psi_2\rangle|^2$ as well as $\langle\psi_1|(P_1 + P_2)|\psi_1\rangle$ are always nonnegative. This completes the proof. □

Theorem[1] establishes that even though a d dimensional quantum system contains uncountable pure states at most d of them can be mutually orthogonally with each other and thus perfectly distinguishable.

We now move to another well established result by Wootters and Zurek[18].

The No Cloning Theorem

Theorem 2. *Unknown quantum states cannot be cloned.*

Proof. We start by assuming that a universal cloning machine exists. This cloning machine can be modelled as a Unitary operator which takes an unknown quantum state $|\psi\rangle$ as input along with a blank state $|B\rangle$ and outputs two copies of the state $|\psi\rangle$. Thus we have

$$U(|\psi\rangle \otimes |B\rangle) = |\psi\rangle \otimes |\psi\rangle \ \forall \ |\psi\rangle \tag{2.5}$$

Now consider a quantum state $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ created from equal superposition of two orthogonal states $|0\rangle$ and $|1\rangle$. From linearity of U, have

$$\begin{aligned} U(|+\rangle \otimes |B\rangle) &= \frac{1}{\sqrt{2}}[U(|0\rangle \otimes |B\rangle) + U(|1\rangle \otimes |B\rangle)] \\ &= \frac{1}{\sqrt{2}}[|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle] \neq |+\rangle \otimes |+\rangle \end{aligned} \tag{2.6}$$

As we arrive at a contradiction such a cloning machine U does not exist. This completes the proof. □

Theorem [1] and Theorem [2], despite appearing entirely unrelated, are in fact equivalent. If it were possible to clone non-orthogonal quantum states, one could distinguish them by generating arbitrary copies of the given state and performing measurements. Conversely, if distinguishing non-orthogonal quantum states were feasible, one could clone them by first identifying the state and subsequently preparing an arbitrary number of identical copies.
In the following subsection, we examine the simplest finite-dimensional quantum system: the qubit.

2.1.3 The qubit

The simplest quantum system corresponds to a two-dimensional complex Hilbert space, denoted as $\mathbb{C}^2$. In this section, we formally derive the most general state for such a system. The dimension of a quantum system refers to the dimension of its associated Hilbert space. Since the density operator for a qubit acts on $\mathbb{C}^2$, it is naturally represented as a 2×2 complex matrix. The Pauli basis of operators plays a crucial role in expressing any such operator efficiently:

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Here, σ^0 represents the identity operator $\mathbb{I}$, while $\{\sigma^i\}_{i=1}^3$ correspond to the Pauli operators, often denoted as σ_x, σ_y and σ_z. (The i appearing in σ^2 is the square root of -1 and shouldn't be confused with i used to refer the pauli matrices. In most cases we simply use σ^2 instead of explicitly writing the matrix so this confusion does not arise.) The traceless Pauli operators $\{\sigma^i\}_{i=1}^3$ will be labeled using lowercase English letters such as $i, j, k \cdots$. In contrast, the full set of four Pauli operators $\{\sigma^\mu\}_{\mu=0}^3$, including the identity, will be indexed using Greek letters such as $\mu, \nu, \alpha, \beta, \cdots$, following a notation convention analogous to the Einstein convention in special and general relativity.

The most general density operator ρ for a qubit system can be expressed as

$$\rho = \frac{1}{2} \sum_{\mu=0}^{3} n_\mu \sigma^\mu, \quad \text{with} \quad n_\mu \in \mathbb{C}. \tag{2.7}$$

The factor of $\frac{1}{2}$ is introduced for notational convenience. By imposing the constraints $\rho = \rho^\dagger$ (hermiticity) and $\mathrm{Tr}[\rho] = 1$ (unit trace), we obtain $n_0 = 1$ and $n_i \in \mathbb{R}$ for all i. Furthermore, to ensure the positivity of ρ, we evaluate its eigenvalues and impose that they must be non-negative. The eigenvalues of ρ are given by

$$\lambda_\pm = \frac{1}{2}\left[1 \pm \sqrt{n_1^2 + n_2^2 + n_3^2}\right]. \tag{2.8}$$

For convenience, we define a real vector $\vec{n} := (n_1, n_2, n_3)^T \in \mathbb{R}^3$. Imposing the positivity condition $\lambda_\pm \geq 0$, we obtain $\sqrt{n_1^2 + n_2^2 + n_3^2} = |\vec{n}| \leq 1$. This provides a useful geometric interpretation of the state space of a qubit system. All vectors $\vec{n} \in \mathbb{R}^3$ satisfying $|\vec{n}| \leq 1$ lie within a ball centered at the origin $(0,0,0)^T$, which is formally known as the *Bloch ball*. The vector $\vec{n}$ is referred to as the *Bloch vector* of the state. The density matrix can then be rewritten in terms of the Bloch vector as

$$\rho_{\vec{n}} = \frac{1}{2}\left[\mathbb{I} + \vec{n} \cdot \vec{\sigma}\right], \quad \text{with} \quad \vec{n} \cdot \vec{\sigma} = \sum_{i=1}^{3} n_i \sigma^i. \tag{2.9}$$

Thus a unit radius ball in $\mathbb{R}^3$ centered at origin serves as an important geometric representation of the state space of a qubit system.

2.1.4 Composite Quantum Systems and the concept of entanglement

Another striking and counterintuitive phenomenon in quantum theory is quantum entanglement. Consider two quantum systems, denoted as S_1 and S_2, with their respective Hilbert spaces H_1 and H_2. According to Postulate IV of quantum mechanics, the Hilbert space associated with the composite system is given by $H_{\text{comp}} = H_1 \otimes H_2$. The postulates of quantum mechanics imply that there exist quantum states in H_{comp} that cannot be expressed as a tensor product of a state in H_1 and a state in H_2. This phenomenon, known as quantum entanglement, signifies a nontrivial correlation between subsystems that has no classical analog. To formalize this concept, we introduce the notion of a separable state.

Definition 6. ***Separable state:*** *A density operator ρ_{12} acting on $H_{comp} = H_1 \otimes H_2$ is called Separable if it can be expressed as a convex mixture of tensor product states in H_1 and H_2. Mathematically*

$$\rho_{12} = \sum_i p_i \, \rho_1^i \otimes \rho_2^i \quad with p_i > 0 \; \forall \; i \; and \; \sum_i p_i = 1 \tag{2.10}$$

where ρ_1^i and ρ_2^i are density operators acting on H_1 and H_2 respectively.

The subscripts appearing under the operators will be used to denote which Hilbert space they act on. Here ρ_{12} denotes that it acts on the composite Hilbert space. Now we are in a position to define the notion of an entangled state:

Definition 7. ***Entangled state:*** *A density operator ρ_{12} acting on $H_{comp} = H_1 \otimes H_2$ is said to be entangled between the Hilbert spaces H_1 and H_2 if it cannot be expressed in a Separable form mentioned in Eq.(2.10).*

Examples of entangled states

The most common examples of entangled states that one might come across are the well known Bell states. The Bell states is the simplest form of pure entanglement between two qubit systems. They are mentioned below:

$$|\beta^{00}\rangle := |\phi^+\rangle = \frac{1}{\sqrt{2}}[|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle] \tag{2.11}$$

$$|\beta^{01}\rangle := |\phi^-\rangle = \frac{1}{\sqrt{2}}[|0\rangle \otimes |0\rangle - |1\rangle \otimes |1\rangle] \tag{2.12}$$

$$|\beta^{10}\rangle := |\psi^+\rangle = \frac{1}{\sqrt{2}}[|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle] \tag{2.13}$$

$$|\beta^{11}\rangle := |\psi^-\rangle = \frac{1}{\sqrt{2}}[|0\rangle \otimes |1\rangle - |0\rangle \otimes |1\rangle] \tag{2.14}$$

The above four Bell states (sometimes also refereed to as maximally entangled states) form a orthonormal basis for the two qubit Hilbert space and thus the set of projectors $\{|\beta^{x_1x_2}\rangle\langle\beta^{x_1x_2}|\}_{x_1,x_2\in\{0,1\}}$ also constitutes a projective measurement on the two qubit Hilbert space. The Bell Basis measurement plays a crucial role in information theoretic tasks such as Superdense coding[19] and Quantum teleportation[20] which is discussed in subsequent subsection[2.1.6].

2.1.5 Holevo's theorem

Holevo's theorem is a fundamental result in quantum information theory, a field at the intersection of physics and computer science. Often referred to as Holevo's bound, it establishes an upper limit on the amount of classical information that can be extracted from a quantum system, known as accessible information. This theorem was first formulated by Alexander Holevo in 1973[21].

Before moving to Holevo's theorem we first define some preliminary quantities from classical information theory namely the Shannon entropy and Mutual Information.

Definition 8. ***Shannon entropy:*** *The Shannon entropy of a discrete classical random variable $x \in X$ generated with probability distribution $\{p(x)\}$ is defined as*

$$H(X) = -\sum_{x\in X} p(x)\log p(x)$$

It quantifies the average amount of uncertainty or information contained in the distribution of X.

Definition 9. ***Mutual Information:*** *The mutual information between two discrete random variables X and Y with joint probability distribution $p(x,y)$ is defined as*

$$I(X:Y) = H(X) + H(Y) - H(X,Y),$$

where $H(X)$ and $H(Y)$ are the Shannon entropies of X and Y, and $H(X,Y)$ is the joint entropy. The mutual information quantifies the amount of information shared between X and Y, measuring how much knowing one variable reduces uncertainty about the other.

Now we are in a position to state the statement of Holevo's theorem.

Theorem 3. *Let X be a classical random variable with probability distribution $\{p(x)\}$, and let $\{\rho_x\}$ be a set of quantum states associated with X such that the ensemble is given by*

$$\rho = \sum_x p(x)\rho_x$$

If a measurement is performed on ρ to extract classical information about X, then the mutual information between the classical variable X and the measurement outcome Y satisfies the bound

$$I(X:Y) \leq S(\rho) - \sum_x p(x)S(\rho_x),$$

where $S(\rho) = -\mathrm{Tr}(\rho \log \rho)$ denotes the von Neumann entropy.

The proof of Holevo's theorem is not included here; however, due to its fundamental importance, the theorem is stated in this thesis. The Holevo bound asserts that while n qubits can encode a larger amount of classical information due to quantum superposition, the maximum amount of classical information that can be extracted through measurement is strictly limited to at most n classical bits.

2.1.6 Superdense coding and Quantum teleportation

Superdense coding

As previously mentioned, a single qubit can transmit an unbounded amount of classical information due to quantum superposition, though this information cannot be reliably decoded. However, if the sender and receiver share a pre-established Bell state, such as $|\phi^+\rangle$, it can be demonstrated that the sender can transmit two classical bits of information by physically communicating only one qubit. This counterintuitive phenomenon, which has no classical counterpart, is aptly referred to as superdense coding [19] devised by Bennett and Wiesner in 1992. The protocol is as follows:

Step 0: The sender say (Alice) and the reciever say (Bob) start by sharing a two qubit state $|\phi^+\rangle_{AB}$. The suffixes A and B denote the qubit held by Alice and Bob respectively.

Step 1: If Alice wishes to send two bits of classical information $(x_1, x_2) \in \{0,1\}^{\times 2}$ to Bob she applies a suitable unitary operation $U^{x_1x_2}$ on her qubit. These four unitaries are taken as$\{U^{00} = \sigma^0, U^{01} = \sigma^3, U^{10} = \sigma^1, U^{11} = \sigma^2\}$. As Bob does nothing in this step his action can be denoted as a $\mathbb{I}$ unitary on his qubit. The updated state between Alice and Bob after application of this unitary is given by:

$$U_A^{x_1x_2} \otimes \mathbb{I}_B \, |\phi^+\rangle_{AB} = |\beta^{x_1x_2}\rangle_{AB} \tag{2.15}$$

Step 3: Next Alice physically sends her qubit to Bob's laboratory.

Step 4: Bob now performs the Bell Basis Projective measurement on the two qubits given by $\{P_{AB}^{y_1y_2} = |\beta^{y_1y_2}\rangle_{AB} \langle\beta^{y_1y_2}|\}_{y_1,y_2\in\{0,1\}}$. This measurement is possible for Bob to implement as he has access to both the qubits in his lab.

Step 5: The probability that Bob gets the outcome y_1y_2 given that Alice applied the unitary $U^{x_1x_2}$ is given by:

$$p(y_1y_2|x_1x_2) = \mathrm{Tr}[P_{AB}^{y_1y_2} \, |\beta^{x_1x_2}\rangle_{AB} \langle\beta^{x_1x_2}|] = \delta_{x_1y_1}\delta_{x_2y_2} \tag{2.16}$$

It is clear that Bob's measurement outcome will satisfy $y_1y_2 = x_1x_2$ and hence Alice has communicated two bits of classical information successfully.

This represents a significant shift in perspective from classical physics, as it demonstrates that qubit communication, when supplemented with preshared entanglement, can double its classical information-carrying capacity. At this stage, one might critique this result by suggesting that the preshared entanglement itself is responsible for the increased communication capacity—perhaps entanglement alone possesses the ability to transmit classical information. However, this is not the case. As we will explore in Section [2.3.1], despite being a highly nontrivial resource, entanglement has no intrinsic communication utility. This further underscores its peculiar nature: while entanglement alone cannot facilitate communication, it can enhance qubit communication when shared between the sender and receiver. Next we demonstrate the phenomenon of Quantum teleportation which is very closely related to superdense coding.

Quantum teleportation

Imagine that a sender, Alice, wishes to physically prepare a pure qubit state $|\psi\rangle \in \mathbb{C}^2$ in the receiver's (Bob's) laboratory. A straightforward approach would be for Alice to prepare the qubit state $|\psi\rangle$ and physically transmit it to Bob, which necessitates the availability of a quantum communication channel. While this method is effective, we consider the scenario where such quantum communication is not available.

To circumvent this limitation, Alice might attempt to send classical information regarding the Bloch vector representation of $|\psi\rangle$, allowing Bob to reconstruct the state. However, this approach requires Alice to communicate an unbounded amount of classical bits, as the Bloch vector can take infinitely many possible values. While unbounded classical communication is impractical, Bob can still prepare approximate states close to $|\psi\rangle$ with finite classical communication, and increasing the amount of classical communication improves the precision of his approximation. This task is widely known as *remote state preparation*, which will be discussed in greater detail in Chapter [6].

To formally define the task of quantum teleportation, let us introduce a third party, Charlie, who wishes to prepare a pure qubit state $|\psi\rangle \in \mathbb{C}^2$ in Bob's laboratory. However, Charlie is unable to communicate with Bob—either classically or quantumly—but is aware that Alice has the capability to do so. Crucially, Charlie does not wish to disclose the identity of $|\psi\rangle$ to Alice. Instead, relying on Holevo's theorem [3], Charlie prepares the state $|\psi\rangle$ and sends the qubit to Alice. Since Alice cannot perfectly determine an arbitrary quantum state from a single copy, she can extract at most one bit of information about $|\psi\rangle$. Any attempt to measure the state to gain information would disrupt it, preventing successful state preparation in Bob's lab. Furthermore, as Alice lacks access to a quantum communication channel, directly forwarding the qubit to Bob is not an option. Under these conditions, it appears that Alice is unable to successfully transmit the state.

This is where the peculiar properties of quantum entanglement become crucial. If Alice and Bob share a maximally entangled Bell state, say $|\phi^+\rangle$, it becomes possible for Alice to implement a protocol that enables Bob to reconstruct $|\psi\rangle$ with only two bits of classical communication. Remarkably, as intended by Charlie, Alice remains completely ignorant of the identity of $|\psi\rangle$ if she follows the protocol to perfectly reconstruct the state in Bob's lab. This groundbreaking protocol was introduced in 1993 by Bennett, Brassard, Crepeau, Jozsa, Peres, and Wootters in [20]. The protocol proceeds as follows:

Step 0: Alice (sender) and Bob (receiver) begin by sharing a maximally entangled two-qubit state $|\phi^+\rangle_{AB}$. The qubit state given to Alice by Charlie is denoted as $|\psi\rangle_C$.

Step 1: After receiving the qubit from Charlie, Alice performs a Bell basis measurement on qubits A and C.

Step 2: The post-measurement three-qubit state, corresponding to the four possible measurement outcomes $\{|\phi^\pm\rangle\langle\phi^\pm|, |\psi^\pm\rangle\langle\psi^\pm|\}$, is given by:

$$|\phi^+\rangle\langle\phi^+| := |\phi^+\rangle_{AC} \otimes \sigma^0 |\psi\rangle_B, \;\; |\phi^-\rangle\langle\phi^-| := |\phi^-\rangle_{AC} \otimes \sigma^3 |\psi\rangle_B,$$
$$|\psi^+\rangle\langle\psi^+| := |\psi^+\rangle_{AC} \otimes \sigma^1 |\psi\rangle_B, \; |\psi^-\rangle\langle\psi^-| := |\psi^-\rangle_{AC} \otimes \sigma^2 |\psi\rangle_B.$$

Step 3: Alice communicates the measurement outcome to Bob using two classical bits.

Step 4: Since the state at Bob's end is always a Pauli-rotated version of $|\psi\rangle$, Bob, upon receiving Alice's message, applies the corresponding Pauli correction unitary σ^μ to recover the state $|\psi\rangle_B$ in his laboratory.

As we can see, Alice gains no knowledge about the state $|\psi\rangle$, ensuring that the protocol adheres to Charlie's requirement of secrecy. This seminal result demonstrates that although two bits of classical communication from Alice to Bob can only convey one bit of classical information about $|\psi\rangle$, the presence of pre-shared entanglement enables Alice to perfectly recreate the quantum state in Bob's laboratory—even when the identity of $|\psi\rangle$ is unknown to her.

2.1.7 Open system dynamics via Quantum Channels

Postulate II describes the unitary evolution of closed, isolated quantum systems. However, most physical systems inevitably interact with their environment, necessitating a framework for understanding state transformations in such scenarios. This study falls under the domain of *open quantum system dynamics*, where these transformations are mathematically modeled as quantum channels acting on the system's state. We begin by introducing relevant notations and preliminary definitions, which will be used extensively throughout this thesis.

- $\mathscr{L}(H)$: The set of all linear operators acting on H.
- $\mathscr{B}(H)$: The set of all bounded linear operators acting on H.
- $\mathscr{H}(H)$: The set of all bounded Hermitian operators acting on H.
- $\mathscr{P}(H)$: The set of all bounded positive operators acting on H.
- $\mathscr{D}(H)$: The set of all density operators acting on H.

For any Hilbert space H, the following strict set inclusion relation holds:

$$\mathscr{D}(H) \subset \mathscr{P}(H) \subset \mathscr{H}(H) \subset \mathscr{B}(H) \subset \mathscr{L}(H). \tag{2.17}$$

In most applications, we focus on the set of bounded operators.

Definition 10. ***Linear map:*** *A map* $\Lambda : \mathscr{B}(H_I) \to \mathscr{B}(H_O)$ *is called linear if, for all* $B_1, B_2 \in \mathscr{B}(H_I)$ *and* $c_1, c_2 \in \mathbb{C}$,

$$\Lambda(c_1 B_1 + c_2 B_2) = c_1 \Lambda(B_1) + c_2 \Lambda(B_2). \tag{2.18}$$

The indices I and O represent the input and output Hilbert spaces associated to Λ. As we are primarily interested in transformations that preserve quantum states. Not all linear maps correspond to physically realizable state transformations, as a state ρ could be mapped to an operator that is not positive. To address this, we introduce the notion of positive maps.

Definition 11. ***Positive map:*** *A linear map* $\Lambda : \mathscr{B}(H_I) \to \mathscr{B}(H_O)$ *is called positive if*

$$\Lambda(P) \in \mathscr{P}(H_O) \quad \forall P \in \mathscr{P}(H_I). \tag{2.19}$$

While positive maps ensure that positive operators remain positive, an unexpected phenomenon arises when applying a positive map to part of an entangled state: some positive maps may yield non-positive operators. Since practical quantum processes often involve transformations on subsystems of entangled states, this behavior is unphysical. To resolve this, we define completely positive (CP)maps.

Definition 12. ***Completely positive map:*** *A linear map* $\Lambda : \mathscr{B}(H_I) \to \mathscr{B}(H_O)$ *is said to be completely positive if, for any auxiliary Hilbert space* H_{aux}*, the extended* *map $id_{aux} \otimes \Lambda$ is also positive, i.e.*

$$id_{aux} \otimes \Lambda(P) \in \mathscr{P}(H_{aux} \otimes H_O) \quad \forall P \in \mathscr{P}(H_{aux} \otimes H_I) \tag{2.20}$$

where $id_{aux} : \mathscr{B}(H_{aux}) \to \mathscr{B}(H_{aux})$ denotes the identity map.

Finally, we require that normalized quantum states remain normalized under transformations. This leads to the definition of completely positive trace-preserving (CPTP) maps, also known as *quantum channels*.

Definition 13. ***Trace preserving map:*** *A linear map is called trace preserving if*

$$\mathrm{Tr}[\Lambda(B)] = \mathrm{Tr}[B] \;\forall\; B \;\in \mathscr{B}(H_I). \tag{2.21}$$

Definition 14. ***Quantum Channel:*** *A quantum channel is a completely positive map $\Lambda : \mathscr{B}(H_I) \to \mathscr{B}(H_O)$ which is also trace preserving.*

The trace-preserving condition along with complete positivity ensures that quantum states remain valid quantum states after transformation and thus quantum channels can be regarded as physical transformations of quantum states.

Kraus operator representation of a Channel

The Kraus operator representation first noted by Sudarshan[22] and later also by Kraus [23] provides a general mathematical framework for describing the evolution of quantum states under quantum channels. Given a quantum channel $\Lambda : \mathscr{B}(H_I) \to \mathscr{B}(H_O)$, its action on a density matrix ρ can be expressed as

$$\Lambda(\rho) = \sum_i K_i \rho K_i^\dagger, \tag{2.22}$$

where $\{K_i\}$ are the Kraus operators satisfying the completeness relation

$$\sum_i K_i^\dagger K_i = \mathbb{I}_I. \tag{2.23}$$

here $\mathbb{I}_I$ denotes the identity operator acting on H_I. The completeness condition ensures trace preservation of the quantum state. The Kraus representation is widely used in quantum information theory to model decoherence, noise, and generalized measurements. It captures all physically valid quantum operations, including unitary evolution, projective measurements, and open-system dynamics induced by interaction with an environment.

Stinespring representation of a Channel

The Stinespring representation provides an isometric description for quantum channels. It states that any quantum channel $\Lambda : \mathscr{B}(H_I) \to \mathscr{B}(H_O)$ can be realized as

$$\Lambda(\rho) = \mathrm{Tr}_E \left[V \rho V^\dagger \right], \tag{2.24}$$

where $V : H_I \to H_O \otimes H_E$ is an isometry satisfying $V^\dagger V = \mathbb{I}_I$, and Tr_E denotes the partial trace over an auxiliary Hilbert space H_E (the environment). This representation models quantum channels as arising from unitary evolution on a larger system followed by discarding environmental degrees of freedom. The Stinespring representation is instrumental in studying noise, quantum error correction, and purification protocols in quantum information theory.

Choi Jamiolkowski Isomorphism

Choi [24] and Jamiolkowski [25] established a one-to-one correspondence between a linear map $\Lambda : \mathscr{B}(H_I) \to \mathscr{B}(H_O)$ and operators acting on the composite Hilbert space $H_I \otimes H_O$. More formally, given any linear map, we can define its Choi matrix J_Λ as

$$J_\Lambda = id \otimes \Lambda(\Phi^+), \tag{2.25}$$

where Φ^+ denotes the unnormalized projector onto the maximally entangled state, given by

$$\Phi^+ = \sum_{i,j=0}^{d_I-1} |i\rangle \langle j| \otimes |i\rangle \langle j|, \tag{2.26}$$

with $\{|i\rangle\}_{i=0}^{d_I-1}$ being an orthonormal basis of H_I (where d_I is the dimension of H_I). The map id in Eq.(2.25) acts on a Hilbert space isomorphic to H_I. Since Eq. (2.25) represents an isomorphism, it is possible to reconstruct the linear map from its Choi matrix. This reconstruction is given by:

$$\Lambda(B) = \mathrm{Tr}_I \left[(B^T \otimes \mathbb{I}) J_\Lambda \right]. \tag{2.27}$$

To better understand the underlying mathematical structure of the Choi-Jamiolkowski (CJ) isomorphism, we first introduce the following notations:

- $\text{LM}(H_I \to H_O)$: The set of all linear maps $\Lambda : \mathscr{B}(H_I) \to \mathscr{B}(H_O)$.
- $\text{PM}(H_I \to H_O)$: The set of all positive maps $\Lambda : \mathscr{B}(H_I) \to \mathscr{B}(H_O)$.
- $\text{CPM}(H_I \to H_O)$: The set of all completely positive maps $\Lambda : \mathscr{B}(H_I) \to \mathscr{B}(H_O)$.
- $\text{TPM}(H_I \to H_O)$: The set of all trace-preserving maps $\Lambda : \mathscr{B}(H_I) \to \mathscr{B}(H_O)$.
- $\text{QC}(H_I \to H_O)$: The set of all quantum channels $\Lambda : \mathscr{B}(H_I) \to \mathscr{B}(H_O)$.

The strength of the CJ isomorphism lies in its persistence across all the sets mentioned above. While we do not introduce all the isomorphic sets here, it is noteworthy that the CJ isomorphism also establishes a one-to-one correspondence between $\text{CPM}(H_I \to H_O)$ and $\mathscr{P}(H_I \otimes H_O)$. Additionally, the trace-preserving condition can be expressed succinctly for the Choi matrix as

$$\text{Tr}_O[J_\Lambda] = \mathbb{I}_I. \tag{2.28}$$

Thus, we arrive at an alternative definition of a quantum channel as follows:

Definition 15. ***Quantum Channel:*** *A quantum channel is a linear map* $\Lambda : \mathscr{B}(H_I) \to \mathscr{B}(H_O)$ *whose Choi matrix* J_Λ *satisfies*

$$J_\Lambda \geq 0 \quad \textit{and} \quad \text{Tr}_O[J_\Lambda] = \mathbb{I}_I. \tag{2.29}$$

With these tools at hand, we now introduce the concept of quantum instruments, which represent more general quantum operations than quantum channels.

2.1.8 Quantum Instruments

As previously discussed, a quantum channel represents the most general transformation of a quantum state. However, in practical scenarios, one may require a transformation that not only evolves a quantum state but also yields a classical outcome. For instance, as outlined in Postulate III, a quantum measurement necessitates describing both the classical outcome and the corresponding post-measurement quantum state. These measurement operations are specific instances of a more general concept known as a *Quantum Instrument*. A quantum instrument permits the coarse-graining of some classical outcomes while preserving the distinction of others. In the extreme case where all outcomes are coarse-grained, the instrument reduces to a quantum channel.

Mathematically, a quantum instrument, denoted by the bold symbol **I**, is defined as follows:

Definition 16. ***Quantum Instrument:*** *A quantum instrument* **I** *is a collection of completely positive (CP) maps* $\{\Lambda_m\}$ *such that their sum constitutes a quantum channel:*

$$\mathbf{I} = \left\{ \Lambda_m \mid \Lambda_m \in CPM(H_I \to H_O),\ with\ \sum_m \Lambda_m = \Lambda \in QC(H_I \to H_O) \right\}, \tag{2.30}$$

where m *labels the classical outcome associated with the quantum instrument. The probability of obtaining the classical outcome* m *given an input state* ρ *is determined by:*

$$p(m|\rho) = \mathrm{Tr}[\Lambda_m(\rho)], \quad \forall \rho \in \mathscr{D}(H_I). \tag{2.31}$$

Since $\sum_m p(m|\rho) = 1$ *for all* ρ*, it follows that the sum* $\sum_m \Lambda_m = \Lambda$ *must define a quantum channel. The corresponding post-measurement quantum state for outcome* m *is given by:*

$$\rho_m = \frac{\Lambda_m(\rho)}{p(m|\rho)} \in \mathscr{D}(H_O), \quad \forall m, \rho. \tag{2.32}$$

To elucidate the distinction between quantum measurements, quantum channels, and quantum instruments, we consider the following special cases:

Special Case (a): The requirement that $\Lambda = \sum_m \Lambda_m$ is a completely positive trace-preserving (CPTP) map imposes the condition that each individual Λ_m must be trace-non-increasing, meaning that:

$$\mathrm{Tr}[\Lambda_m(\rho)] = p(m|\rho) \leq \mathrm{Tr}[\rho] = 1, \quad \forall \rho \in \mathscr{D}(H_I). \tag{2.33}$$

If each CP map Λ_m admits a Kraus representation of the form $\Lambda_m(\rho) = M_m \rho M_m^\dagger$, then the trace-non-increasing constraint requires that the operators M_m satisfy $M_m^\dagger M_m \leq \mathbb{I}_I$. This formulation corresponds precisely to the notion of a quantum measurement, as described in Postulate III.

Special Case (b): If the quantum instrument has only a single outcome (sometimes referred to as a *trivial* outcome), it effectively reduces to a quantum channel.

Thus, a general quantum instrument $\mathbf{I} = \{\Lambda_m\}$ can be interpreted as a quantum measurement wherein certain classical outcomes are coarse-grained. Mathematically, each Λ_m admits a Kraus representation:

$$\Lambda_m(\rho) = \sum_{i_m} K_{i_m,m} \rho K_{i_m,m}^\dagger, \quad \text{with} \quad \sum_{i_m} K_{i_m,m}^\dagger K_{i_m,m} \leq \mathbb{I}_I. \tag{2.34}$$

The overall action of the quantum channel is then given by:

$$\Lambda(\rho) = \sum_m \Lambda_m(\rho) = \sum_m \sum_{i_m} K_{i_m,m} \rho K_{i_m,m}^\dagger, \quad \text{with} \quad \sum_m \sum_{i_m} K_{i_m,m}^\dagger K_{i_m,m} = \mathbb{I}_I. \tag{2.35}$$

The concept of quantum instruments is fundamental in the formulation of higher-order quantum theories, as discussed in Section [2.4].

2.2 General Probabilistic Theories

General Probabilistic Theories (GPTs) constitute a broad theoretical framework designed to encompass a wide variety of probabilistic models, including both classical and quantum theories as special cases. This framework allows for the formulation of mathematically consistent theories that, in certain instances, exhibit post-quantum phenomena. Some of these post-quantum effects lead to counterintuitive or physically implausible consequences, thereby highlighting the distinctive characteristics of quantum mechanics that underpin its remarkable success in describing physical reality. Consequently, GPTs facilitate a deeper understanding of the fundamental principles governing physical theories by delineating the phenomena that are theoretically admissible and those that are not expected to manifest in nature.

The GPT framework is sufficiently general to encompass all conceivable probabilistic theories that employ states to determine measurement outcome probabilities. Although the origins of this framework can be traced back to the 1960s [26, 27], it has recently gained renewed interest, particularly within the field of quantum information theory [28–31]. Within this framework, a physical system S is characterized by a triplet $(\Omega_S, \mathscr{E}_S, T_S)$, which respectively denotes its state space, effect space, and transformation space.

2.2.1 Fundamental Components of GPTs

A GPT begins by defining an abstract system of interest, denoted as S. It is postulated that any physical system must possess at least three fundamental components:

- **State Space (Ω_S):** The set of all possible preparations of the physical system.
- **Effect Space ($\mathscr{E}_S$):** The set of all measurement effects that can be observed in an experimental setting when interacting with the system.
- **Transformation Space (T_S):** The set of all valid transformations that map any initial state to another permissible state within the theory.

Thus, any system S is fully specified by the triplet representation $S := (\Omega_S, \mathscr{E}_S, T_S)$. In the following sections, we provide a more detailed exposition of these fundamental components.

2.2.2 States, Effects and Transformations

We now formally define the three sets $(\Omega_S, \mathscr{E}_S, T_S)$.

Definition 17. ***State Space:*** *For a given system S, Ω_S denotes the set of normalized states of the system, where a state is a mathematical object yielding outcome probabilities for all measurements that can be performed on the system. Generally, Ω_S is considered to be a compact convex set embedded in some finite dimensional real vector space V, as it is natural to assume that if two states ω and ω' in Ω_S can be prepared, then any arbitrary convex mixture $p\omega + (1-p)\omega'$ can also be prepared. The extreme points of this set are called pure states.*

Definition 18. ***Effect Space:*** *An effect e is a map $e : \Omega_S \to [0,1]$, where $e(\omega)$ denotes the probability of obtaining the outcome corresponding to effect e when a measurement is performed on the state $\omega \in \Omega_S$. Effects are considered to be linear functionals, satisfying $e(p\omega + (1-p)\omega') = pe(\omega) + (1-p)e(\omega')$ for all $\omega, \omega' \in \Omega_S$ and $p \in [0,1]$. A special effect, called the unit effect u, is defined as $u(\omega) = 1$ for all $\omega \in \Omega_S$. The set of all proper effects is denoted as $\mathscr{E}_S \equiv \{e \mid 0 \le e(\omega) \le 1, \ \forall\ \omega \in \Omega_S\}$. The effect space is embedded in the dual vector space $V^\star$. A measurement $\mathbf{M}$ is a collection of effects summing to the unit effect, i.e., $\mathbf{M} \equiv \{e_i \in \mathscr{E}_S \mid \sum_i e_i = u\}$.*

Definition 19. ***State and Effect Cones:*** *It is sometimes convenient to work with unnormalized states and effects. The set of unnormalized states forms a cone $V_+ \subset V$, where $V_+ = \{r\omega \mid \omega \in \Omega_S, r \ge 0\}$. The set of unnormalized effects forms a dual cone $V_+^\star \subset V^\star$, where $V_+^\star \equiv \{e \mid e(\omega) \ge 0, \ \forall\ \omega \in \Omega_S\}$. In this representation, extreme effects can be further classified as ray extremal and non-ray extremal effects. An unnormalized effect $e \in V_+^\star$ is ray extremal if it cannot be written as* *a conical mixture of other rays, i.e.,* $e = \mu_0 e_0 + \mu_1 e_1$ *with* $e_0, e_1 \in V_+^\star$ *and* $\mu_0, \mu_1 > 0$ *implies* $e_0 = e_1 = e$. *We generally assume the 'no-restriction hypothesis,' which states that the set of possible measurements is the dual of the set of states [32].*

Definition 20. ***Transformation Space:*** *The set* T_S *consists of transformations that map states to states, i.e.,* $T(V_+) \subset V_+$ *for all* $T \in T_S$. *These transformations are assumed to be linear to preserve statistical mixtures, i.e.,* $T(p\omega + (1-p)\omega') = pT(\omega) + (1-p)T(\omega')$. *We also demand* T_S *consists only of normalization-preserving transformations, which satisfy* $T(\Omega_S) \subset \Omega_S$.

2.2.3 Operational Dimension and Information Dimension

Definition 21. ***Operational Dimension:*** *A set of states* $\{\omega_i\} \subset \Omega$ *is said to be perfectly distinguishable if there exists a measurement* $\mathbf{M} \equiv \{e_i \mid \sum_i e_i = u\}$ *such that* $e_i(\omega_j) = \delta_{ij}$. *Given a system* $S \equiv (\Omega, E)$, *the maximum cardinality of a set of states that can be perfectly distinguished is referred to as the operational dimension of the system. It is denoted by* $OD(S)$.

Definition 22. ***Information Dimension:*** *A set of states* $\{\omega_i\} \subset \Omega$ *is said to be pairwise distinguishable if every pair of states* (ω_i, ω_j) *is distinguishable. Given a system* $S \equiv (\Omega, E)$, *the maximum cardinality of a set of states that can be pairwise distinguished is referred to as the information dimension of the system. It is denoted by* $ID(S)$.

It is straightforward to observe that any set of states that are perfectly distinguishable must also be pairwise distinguishable. Consequently, for any system S, the inequality $ID(S) \geq OD(S)$ holds. In the case where S is a quantum system, any set of pairwise orthogonal states is necessarily mutually orthogonal. Therefore, for quantum systems, the operational and information dimensions are identical are are equal to the dimension of the associated Hilbert space. If for a system S we have $ID(S) > OD(S)$ then S is said to exhibit the phenomenon of *Dimension Mismatch*.

From this point forward, we outline the method for describing the composition of two systems within the GPT framework.

2.2.4 Composite systems

Given two systems A and B, the composite state space Ω_{AB} is embedded within the tensor product space $V_A \otimes V_B$. While the choice of Ω_{AB} is not uniquely determined, it is constrained by the no-signaling principle and the local tomography postulate [28]. The no-signaling principle imposes the requirement that no superluminal influence can exist between the various subsystems of a composite system. On the other hand, the local tomography postulate asserts that, given an arbitrary number of copies of a state, it is possible to fully determine the state by performing local measurements on the subsystems and collecting the resulting data. These constraints restrict the possible choices to two extreme cases: the minimal tensor product space ($\otimes_{\text{min}}$) and the maximal tensor product space ($\otimes_{\text{max}}$) [33]. Formally, these are defined as follows:

Definition 23. ***Minimal tensor product:*** *The minimal tensor product corresponds to a composite state space that includes only separable states as valid states in the composite theory. That is,*

$$\Omega_{AB}^{\text{min}} \equiv \left\{ \omega_{AB} = \sum_i p_i \omega_A^i \otimes \omega_B^i \;\middle|\; \omega_A^i \in \Omega_A,\ \omega_B^i \in \Omega_B,\ p_i \geq 0,\ \sum_i p_i = 1 \right\}. \tag{2.36}$$

The corresponding effect space $\mathscr{E}_{AB}^{\text{min}}$ is determined under the assumption of the no-restriction hypothesis.

Definition 24. ***Maximal tensor product:*** *The maximal tensor product corresponds to a composite effect space that includes only separable effects as valid effects in the composite theory. That is,*

$$\mathscr{E}_{AB}^{\text{max}} \equiv \left\{ e_{AB} = \sum_i p_i e_A^i \otimes e_B^i \;\middle|\; e_A^i \in \mathscr{E}_A,\ e_B^i \in \mathscr{E}_B,\ p_i \geq 0,\ \sum_i p_i = 1 \right\}. \tag{2.37}$$

The corresponding state space Ω_{AB}^{max} is determined under the assumption of the no-restriction hypothesis.

With these structures in place, we are now equipped to formally interpret quantum theory within the framework of Generalized Probabilistic Theories (GPTs). In the next subsection, we proceed with this analysis in detail.

2.2.5 Quantum Theory as a GPT

Consider a finite-dimensional quantum system S_Q associated with a Hilbert space H. Within the framework of Generalized Probabilistic Theories (GPTs), S_Q can be represented as the triplet $(\Omega_{S_Q}, \mathscr{E}_{S_Q}, T_{S_Q})$, defined as follows:

$$\Omega_{S_Q} := \mathscr{D}(H), \quad \mathscr{E}_{S_Q} := \{\pi \mid \pi \in \mathscr{P}(H) \text{ and } \pi \leq \mathbb{I}_H\}, \quad T_{S_Q} := \text{QC}(H \mapsto H). \tag{2.38}$$

As we can see, in certain cases, the set of allowed transformations must be further constrained to ensure that operations acting on one subsystem of an entangled state map to valid quantum states. Notably, both the state and effect cones in quantum theory are given by $\mathscr{P}(H)$, making quantum theory a *self-dual* theory. In the following subsection, we describe how quantum systems compose within the GPT framework.

2.2.6 Composition of Quantum Systems

As previously discussed, the principles of no-signaling and tomographic locality impose constraints on the possible compositions of quantum systems, restricting them to lie between the minimal and maximal tensor product spaces. Here, we analyze the composition of two quantum systems, S_{Q_1} and S_{Q_2}, associated with Hilbert spaces H_1 and H_2, respectively.

Importantly, it is possible to construct compositions of quantum systems that do not retain the self-dual structure of standard quantum theory. The *minimal* tensor product composition in quantum theory corresponds to the *Separable SEP* theory. Conversely, the *maximal* tensor product composition is referred to as the $\overline{SEP}$ theory. These compositions can be formally defined in terms of their associated state and effect spaces:

$$\Omega^{SEP} = \{\rho_{12} \mid \rho_{12} \text{ is a separable density operator} \in \mathscr{D}(H_1 \otimes H_2)\}, \tag{2.39a}$$

$$\mathscr{E}^{SEP} = \left\{\pi_{12} \;\middle|\; 0 \leq \mathrm{Tr}[\pi_{12}\rho_{12}] \leq 1,\ \forall\ \rho_{12} \in \Omega^{SEP}\right\}. \tag{2.39b}$$

$$\mathscr{E}^{\overline{SEP}} = \left\{\pi_{12} \;\middle|\; \pi_{12} = \sum_i \pi_1^i \otimes \pi_2^i,\ \text{where } \pi_1^i, \pi_2^i \geq 0 \text{ and } 0 \leq \pi_{12} \leq \mathbb{I}_{12}\right\}, \tag{2.40a}$$

$$\Omega^{\overline{SEP}} = \left\{\rho_{12} \;\middle|\; 0 \leq \mathrm{Tr}[\pi_{12}\rho_{12}] \leq 1,\ \forall\ \pi_{12} \in \mathscr{E}^{\overline{SEP}}\right\}. \tag{2.40b}$$

It is important to note that the sets $\mathscr{E}^{SEP}$ and $\Omega^{\overline{SEP}}$ include certain non-positive operators. Nevertheless, these operators still yield valid probability values when evaluated on separable states or effects, respectively. The effect cone of $\mathscr{E}^{SEP}$ and the state cone of $\Omega^{\overline{SEP}}$ define a broader class of Hermitian operators that remain positive on product states, commonly referred to as *POPT* (Positive on Product Test) operators.

In Chapter[3], we revisit the composition of quantum systems and analyze the statistical properties emerging from such compositions in greater detail.

2.2.7 Polygon Models

This class of theoretical models was initially introduced in [34] to explore the constraints on statistics imposed by the structure of the local state space. Although these models remain hypothetical, they have been the subject of numerous recent investigations, leading to several nontrivial foundational insights into the mathematical and conceptual framework of Hilbert space quantum theory [35–41]. We denote such a system by $\mathscr{P}_{ly}(n) \equiv (\Omega_n, \mathscr{E}_n, \mathscr{T}_n)$, where n is a positive integer.

State Space:

For each elementary system, the state space Ω_n is represented by a regular n-sided polygon. Equivalently, the state space can be described as the convex hull of n pure states $\{\omega_i\}_{i=1}^n \subset \mathbb{R}^3$, given explicitly by

$$\omega_i = \begin{pmatrix} r_n \cos\frac{2\pi i}{n} \\ r_n \sin\frac{2\pi i}{n} \\ 1 \end{pmatrix}, \text{ where } r_n = \sqrt{\sec(\pi/n)}. \tag{2.41}$$

For $n = 1,2,3$, the state spaces form simplices (classical), while for $n \geq 4$, new structural features emerge. Specifically, for $n \geq 4$, certain mixed states admit non-unique decompositions in terms of extremal points. In the limit where the state space approaches a circular geometry, the pure states take the form $\omega_\theta = (\cos\theta, \sin\theta, 1)^{\mathrm{T}}$, with θ varying continuously in the range $[0, 2\pi]$.

Effect Space:

The effect space is constructed to ensure that the outcome probabilities, defined as $e_i(\omega_j) := e_i^{\mathrm{T}}\omega_j$, remain valid, i.e.,

$$0 \leq e_i(\omega_j) \leq 1, \ \ \forall \omega_j \in \Omega_n, e_i \in \mathscr{E}_n. \tag{2.42}$$

Among the extremal elements of the convex effect space $\mathscr{E}_n$, two effects are uniquely designated across all models: the null effect Θ and the unit effect u, which are defined as

$$\Theta := (0,0,0)^{\mathrm{T}}, \quad u := (0,0,1)^{\mathrm{T}}. \tag{2.43}$$

The remaining extremal effects $\{e_i\}_{i=1}^n$ and their complementary effects $\{\bar{e}_i := u - e_i\}_{i=1}^n$ are given by:

$$e_i = \frac{1}{2}\begin{pmatrix} r_n \cos\frac{(2i-1)\pi}{n} \\ r_n \sin\frac{(2i-1)\pi}{n} \\ 1 \end{pmatrix}, \quad \text{for even } n, \tag{2.44a}$$

$$e_i = \frac{1}{1+r_n^2}\begin{pmatrix} r_n \cos\frac{2i\pi}{n} \\ r_n \sin\frac{2i\pi}{n} \\ 1 \end{pmatrix}, \quad \text{for odd } n. \tag{2.44b}$$

For even-sided polygons, the complementary effects $\bar{e}_i$ are extremal and also ray-extremal effects. However, in the case of odd-sided polygons, while e_i remains a ray-extremal effect, its complement $\bar{e}_i$ does not exhibit ray-extremality. For circular state spaces, the extremal effects are given by $e_i := \frac{1}{2}(\cos\theta_i, \sin\theta_i, 1)^{\mathrm{T}}$, with $\theta_i \in [0, 2\pi]$.

2.3 Spacelike and Timelike Correlation Experiments

When analyzing experiments involving composite systems within the spacetime framework, two fundamentally distinct experimental scenarios naturally arise. Consider a setting in which two agents, Alice and Bob, perform measurements on separate subsystems, each held by one of them. The embedding of these measurements within spacetime can occur in two distinct manners. Either the two measurement actions are spacelike separated (see Fig. [2.1(b)]), implying that no communication is possible between the agents during the experiment, or they are timelike separated (see Fig. [2.1(a)]), where one action occurs strictly in the causal past of the other.

Both of these scenarios have been extensively studied within the information theory community. The statistical correlations observed by the two agents in these settings are commonly referred to as spacelike correlations and timelike correlations, respectively. A seminal result in this context is due to John Bell [1], who demonstrated that nature cannot be described by a local realistic theory, a conclusion derived from an analysis of spacelike correlations.

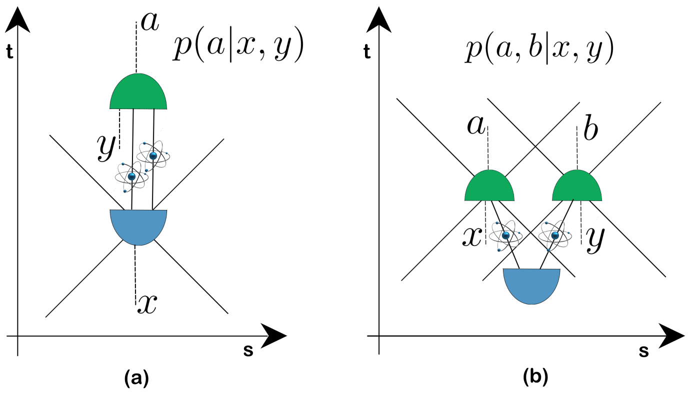

FIG. 2.1 Timelike and spacelike correlations. (a) the preparation (blue) and measurement (green) devices receive inputs x and y, respectively and finally an outcome a is obtained. The correlation $p(a|x,y)$ is called a timelike correlation. Whereas in figure (b) correlations generated by spacelike separated measurement devices acting on separate systems is shown. The correlation $p(ab|xy)$ is called a spacelike correlation

In recent years, the study of timelike correlations has garnered significant attention, as evidenced by works such as [42–46], among others. In this section, we provide a brief overview of some key results pertaining to both spacelike and timelike correlations, as they serve as foundational prerequisites for the work presented in this thesis. We begin with a discussion of the phenomenon of Bell nonlocality.

2.3.1 Bell Nonlocality

The Bohr-Einstein debate was a series of intellectual exchanges between Niels Bohr and Albert Einstein concerning the fundamental nature of quantum mechanics. Einstein, skeptical of quantum indeterminacy, proposed a series of thought experiments challenging the completeness of quantum theory, famously asserting, "God does not play dice." Bohr, in contrast, defended the Copenhagen interpretation, arguing that quantum mechanics is inherently probabilistic and that measurement influences the observed system. Their discussions culminated in the examination of quantum entanglement, leading to the formulation of the Einstein-Podolsky-Rosen(EPR) paradox [47]. This debate laid the groundwork for subsequent developments in quantum information theory and Bell's theorem, which provided an experimental framework to test quantum nonlocality.

Bell's theorem begins with an intuitive assumption about reality, derived from our everyday experiences. As articulated by Bell, this assumption is termed *local causality* or *local realism*. It can be further decomposed into two fundamental premises: realism and locality. To formally define these concepts, we first establish the experimental scenario.

Consider a spacelike-separated experimental setup involving two parties, Alice and Bob. Each independently selects experimental settings, denoted by variables x and y, respectively, assuming complete free will in their choices. The corresponding experimental outcomes are represented by a and b. The observed joint statistics or correlations are described by the probability distribution $\{p(ab|xy)\}$. We introduce a hidden variable λ that deterministically governs the measurement outcomes, assuming it is drawn from a prior distribution $\mu(\lambda)$ over some abstract hidden variable space $\mathbf{L}$.

The assumption of realism, or determinism, is formally stated as follows:

Definition 25. ***Realism:*** *The hidden variable λ fully determines the experimental outcomes of any measurement, i.e.,*

$$p(ab|xy\lambda) \in \{0,1\}, \quad \forall \lambda \in \mathbf{L}, a, b, x, y. \tag{2.45}$$

Furthermore, Bell incorporated the constraint of relativistic causality through the assumption of locality:

Definition 26. ***Locality:*** *The hidden variable λ does not allow information about one experimenter's setting to influence the other experimenter's outcome. Mathematically, the marginal statistics satisfy:*

$$p(a|xy\lambda) = p(a|x\lambda), \quad \forall \lambda, a, x, y, \tag{2.46a}$$

$$p(b|xy\lambda) = p(b|y\lambda), \quad \forall \lambda, b, y, x. \tag{2.46b}$$

These conditions ensure that any correlation $\{p(ab|xy)\}$ satisfying both locality and realism must be expressible in the factorized form:

$$p(ab|xy) = \int_{\lambda \in \mathbf{L}} \mu(\lambda) p(a|x\lambda) p(b|y\lambda) d\lambda. \tag{2.47}$$

Bell demonstrated that quantum mechanics predicts correlations that do not satisfy this factorization, violating the assumptions of local realism. Correlations that violate local realism are referred to as *nonlocal* or *nonfactorizable* correlations.

The 2022 Nobel Prize in Physics was awarded to Alain Aspect, John Clauser, and Anton Zeilinger [2–5] for their experimental confirmation of nonlocal correlations, validating Bell's predictions. In the next section, we discuss one such experimental realization: the XOR game.

The Simplest Bell Scenario and the XOR Game

Any experiment described in the above section is called a *Bell scenario*. The simplest Bell scenario considers binary values for all classical variables x, y, a, and b. This setup can be formulated as a game known as the *XOR game*. Here, the referee provides Alice and Bob with random variables x and y. The referee also ensures that the actions of Alice and Bob are space like separated and thus they cannot communicate. The winning condition requires that their outputs satisfy $a(x) \oplus b(y) = xy$.

For factorizable correlations, it can be shown that the success probability is upper bounded by $3/4$. However, a quantum strategy can violate this bound. The protocol is as follows:

Alice and Bob preshare the Bell state $|\psi^-\rangle$. Their measurement outcomes are given by the POVMs $\pi_A^{a|x}$ and $\pi_B^{b|y}$, defined as:

$$\pi_A^{a|x} = \frac{1}{2}\left[\mathbb{I} - (-1)^a \frac{1}{\sqrt{2}}\left\{\sigma^3 + (-1)^x \sigma^1\right\}\right], \tag{2.48a}$$

$$\pi_B^{b|y} = \frac{1}{2}\left[\mathbb{I} + (-1)^b \left\{\delta_{0y}\sigma^3 + \delta_{1y}\sigma^1\right\}\right]. \tag{2.48b}$$

The resulting correlation is:

$$p(ab|xy) = \frac{1}{4}\left[1 + (-1)^{a\oplus b\oplus xy}\frac{1}{\sqrt{2}}\right]. \tag{2.49}$$

Computing the success probability:

$$P^{success} = \sum_{a\oplus b = xy} \frac{1}{4} p(ab|xy) = \frac{1}{2}\left[1 + \frac{1}{\sqrt{2}}\right] > \frac{1}{2}\left[1 + \frac{1}{2}\right] = \frac{3}{4} \tag{2.50}$$

Thus, quantum correlations violate the bound imposed by local realism, demonstrating quantum nonlocality. Sometimes the XOR game is also presented in form of a linear inequality known as Clauser-Horne-Shimony-Holt *CHSH* Bell inequality [4]. We denote the *CHSH* operator as:

$$\mathbf{B_{CHSH}} = A_0 \otimes B_0 + A_0 \otimes B_1 + A_1 \otimes B_0 - A_1 \otimes B_1 \tag{2.51}$$

where, A_x, B_y denote the Hermitian measurement operators of Alice and Bob. Relabeling the outcomes $a, b \in \{0,1\} \mapsto \{+1,-1\}$ the Bell inequality can also be expressed using the expectation value of $\mathbf{B_{CHSH}}$ as:

$$|\langle \mathbf{B_{CHSH}} \rangle| \leq 2 \tag{2.52}$$

Any local realistic theory must satisfy the above inequality. The maximal obtainable expectation value for the $\mathbf{B_{CHSH}}$ under quantum theory is given by the Tsirelson's bound $2\sqrt{2}$ [48]. Next, we introduce the concept of Hardy's paradox [49].

Hardy Paradox

The Hardy paradox, introduced by Lucien Hardy in 1993, is a thought experiment designed to illustrate quantum nonlocality without relying on inequalities, thereby providing an alternative demonstration to Bell's theorem. This paradox reveals that certain quantum mechanical predictions cannot be reconciled with any local realistic theory.

Consider the following correlation structure within the simplest Bell scenario, characterized by two parties, each with two possible measurement settings and two possible outcomes:

$$P(+1,+1|\, \mathbf{M}_1^A,\ \mathbf{N}_1^B) = p > 0, \tag{2.53a}$$

$$P(+1,+1|\, \mathbf{M}_1^A,\ \mathbf{N}_2^B) = 0, \tag{2.53b}$$

$$P(+1,+1|\, \mathbf{M}_2^A,\ \mathbf{N}_1^B) = 0, \tag{2.53c}$$

$$P(-1,-1|\, \mathbf{M}_2^A,\ \mathbf{N}_2^B) = 0, \tag{2.53d}$$

The outcomes are relabelled as relabeled as $a, b \in \{0,1\} \mapsto \{+1,-1\}$ and $\mathbf{M}_{1/2}^{A/B}$ denote the two measuremts of Alice and Bob. These four conditions cannot simultaneously hold true for any correlation that admits a local or factorizable hidden-variable model. However, quantum mechanics predicts the existence of such correlations, thereby challenging the assumptions of local realism and further substantiating the inherently nonlocal nature of quantum phenomena.

In quantum mechanics the maximum value of Hardy's success probability p has been shown to be $(-11+5\sqrt{5})/2 \approx 0.09$ [50, 51].

2.3.2 Classical simulation of a qubit channel

In this section, we discuss scenarios involving timelike correlations. A classic example of such an experiment is the Holevo scenario (see Subsection [2.1.5]). In this context, it is observed that, although a sender, Alice, can encode an arbitrary amount of classical information into a single qubit, the receiver, Bob, cannot reliably decode more than one classical bit of information.

In the Holevo scenario, the figure of merit is typically the maximum achievable mutual information between Alice and Bob. Holevo's theorem establishes a fundamental upper bound on this mutual information, which cannot be surpassed for qubit communication. However, Frenkel and Weiner, in their work [45], posed a more general question: what if the metric is not constrained to the maximum possible mutual information? Is it possible to evaluate the statistical outcomes of qubit-based protocols on a different basis?

Their findings and implications are discussed in the following.

The Holevo-Frenkel-Weiner Scenario

We begin by considering alternative metrics that may be employed to evaluate the performance of qubit-based communication relative to classical bit-based communication. To explore all such potential metrics, we examine the following scenario.

Assume that Alice encodes classical information x onto a qubit state ρ_x and subsequently transmits it to Bob. Bob then attempts to infer about x by performing a positive operator-valued measure (POVM) with n outcomes, denoted by the set $\mathbf{M} = \{\pi_b \geq 0\}_{b=1}^{n}$, where $\sum_b \pi_b = \mathbb{I}$. The conditional probability distribution governing Bob's measurement outcomes can be expressed as follows:

$$p(b|x) = \mathrm{Tr}[\rho_x \pi_b]. \tag{2.54}$$

The resulting statistics $\{p(b|x)\}$ can be regarded as a classical communication channel with input variable x and output variable b. We now pose the following question: Is it possible for Alice and Bob to simulate any such channel $\{p(b|x)\}$ using a single bit of classical communication supplemented with shared randomness? If this is not feasible, then it becomes evident that one can construct a suitable metric demonstrating the superior communication power of qubits over classical bits. We now present a theorem by Frankel and Weiner that addresses this question.

Theorem 4. *Any classical channel* $\{p(b|x)\}$ *that can be expressed in the form*

$$p(b|x) = \mathrm{Tr}[\rho_x \pi_b], \tag{2.55}$$

where $\rho_x \in \mathscr{D}(\mathbb{C}^d)$ *and* $\pi_b \in \mathscr{P}(\mathbb{C}^d)$ *represent POVM elements summing to identity, can be simulated using shared randomness and a classical communication system with* d *distinguishable pure states.*

The above theorem is not restricted to the qubit case but applies more generally to quantum systems of arbitrary dimension. Nevertheless, it conclusively demonstrates that a single classical bit can replicate the capabilities of a single qubit, provided that shared randomness is treated as a free resource.

Up to this point, our analysis has focused on the measurement statistics of individual qubits. In the following section, we extend our investigation to composite systems, where the validity of Theorem [4] no longer holds. Next, we introduce an experimental scenario in the time-like domain that leverages joint statistics from composite systems to reveal the limitations of classical communication in such contexts.

The Random Access Code Scenario

The Random Access Code (RAC) scenario can be regarded as a generalization of the Holevo-Frenkel-Weiner (HFW) scenario. Here, we assume the presence of a third party, say Charlie, who provides Bob with a classical random variable, denoted as y, which remains unknown to Alice. Bob is tasked with computing the statistics of the composite system state $\rho_x \otimes |y\rangle\langle y|$. The states $|y\rangle$ can be interpreted as orthogonal states in a finite- or infinite-dimensional Hilbert space, denoted as H_C. Defining $\mathbf{M}_C = \{\pi_b\}_{b=1}^n$ as an n-outcome measurement on the composite system, the corresponding statistics are given by

$$p(b|x,y) = \mathrm{Tr}[(\rho_x \otimes |y\rangle\langle y|)\pi_b], \tag{2.56}$$

where ρ_x represents Alice's encoding onto qubit states, and thus the composite state $\rho_x \otimes |y\rangle\langle y| \in \mathscr{D}(\mathbb{C}^2 \otimes H_C)$. An important property of Eq. (2.56) is that it can be equivalently rewritten as

$$p(b|x,y) = \mathrm{Tr}[(\rho_x \otimes |y\rangle\langle y|)\pi_b] = \mathrm{Tr}[\rho_x \pi_{b|y}] \tag{2.57}$$

where $\pi_{b|y} = \text{Tr}_C[(\mathbb{I} \otimes |y\rangle\langle y|)\pi_b] \in \mathscr{P}(\mathbb{C}^2)$ satisfies the completeness relation $\sum_{b=1}^{n} \pi_{b|y} = \mathbb{I} \ \forall \ y$. This formulation implies that for each y, Bob effectively performs a measurement $\mathbf{M}_y = \{\pi_{b|y}\}_{b=1}^{n}$ on ρ_x. Conversely, given an arbitrary set of measurements $\mathbf{M}_y$, possibly with different numbers of outcomes, one can interpret these statistics in terms of composite system measurements with y being classical states.

A crucial observation is that Alice does not possess the knowledge of y, nor does she know which measurement among the set $\{\mathbf{M}_y\}$ Bob will implement on ρ_x. Notably, when this information remains inaccessible to Alice, qubit communication can outperform classical bit communication. To illustrate this phenomenon, consider the well-known $2 \mapsto 1$ RAC task:

$2 \mapsto 1$ **RAC Task:** Alice is given a two-bit string, $x_0x_1 \in \{0,1\}^2$. Here, we assume H_C is two-dimensional, and the states $|y\rangle \in \{|0\rangle, |1\rangle\}$ represent two possible orthogonal classical states provided to Bob. Bob's objective is to return a bit $b = x_y$ when given the state $|y\rangle$. We now demonstrate that qubit communication provides an advantage over classical bit communication.

The performance metric in this scenario is the probability of Bob correctly guessing x_y, given by:

$$P_{\text{succ}}^{\text{RAC}} = \sum_{x_0,x_1,y=0}^{1} \frac{1}{8} p(b = x_y | x_0 x_1 y). \tag{2.58}$$

Since $P_{\text{succ}}^{\text{RAC}}$ is a linear functional of these probabilities, the optimal classical success probability must be achieved via a deterministic encoding and decoding strategy, as shared randomness does not provide any additional advantage. By considering all possible deterministic encoding strategies, it can be shown that the optimal classical success probability is $3/4$.

For the quantum strategy, consider the following protocol:

- Alice encodes the bit string x_0x_1 into a qubit state:

$$\rho_{x_0x_1} = \frac{1}{2}\left[\mathbb{I} + \frac{1}{\sqrt{2}}\left\{(-1)^{x_0}\sigma^1 + (-1)^{x_1}\sigma^3\right\}\right]. \tag{2.59}$$

- Alice sends this state to Bob. Upon receiving it, Bob performs the measurement $\mathbf{M}_y$ corresponding to the state $|y\rangle$ given to him. The measurements are defined as:

$$\mathbf{M}_0 = \left\{ \frac{1}{2}(\mathbb{I} + \sigma^1), \frac{1}{2}(\mathbb{I} - \sigma^1) \right\}, \tag{2.60a}$$

$$\mathbf{M}_1 = \left\{ \frac{1}{2}(\mathbb{I} + \sigma^3), \frac{1}{2}(\mathbb{I} - \sigma^3) \right\}. \tag{2.60b}$$

If the first outcome of the measurement is observed, Bob guesses $b = 0$; otherwise, he guesses $b = 1$.

Substituting this strategy into Eq. (2.58), we obtain the optimal quantum success probability:

$$P_{\text{succ}}^{\text{RAC}} = \frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right) > \frac{3}{4}. \tag{2.61}$$

This result demonstrates the inherent advantage of qubit communication over classical bit communication.

It is important to note that in RAC scenarios, Charlie always provides Bob with orthogonal quantum states $|y\rangle$. A natural generalization involves the case where Charlie provides arbitrary states in H_C. This more general scenario is analyzed in greater detail in Chapter [6] of this thesis.

2.4 Process Matrix framework

In the preceding section, we encountered correlation experiments that can be embedded within spacetime in either a spacelike or timelike configuration. Conventional physical intuition suggests that these two scenarios exhaust the possible causal relations between events localized in spacetime. However, a growing body of research indicates that, if one relinquishes the traditional, globally-defined notion of spacetime and instead considers spacetime structures only locally within distinct laboratories, it becomes possible to obtain novel classes of correlations that defy explanation within the standard spacetime framework. Such correlations are referred to as *indefinite causal order* correlations.

This concept has been introduced and developed in various contexts by different researchers. For instance, Hardy proposed the causaloid framework [9, 10, 52], while Chiribella, D'Ariano, and Perinotti formulated the higher-order maps framework [12, 53]. Another significant contribution came from Oreshkov, Costa, and Brukner, who introduced the process matrix framework [11] (see also [54]).

In this section, we present the mathematical formalism underlying the process matrix framework as introduced by Oreshkov, Costa, and Brukner. First, we discuss a foundational theorem in quantum theory, namely Gleason's theorem [55], which was later generalized by Busch [56]. This theorem plays a key role in the derivation of Process Matrices.

2.4.1 Gleason-Busch theorem

Theorem 5. *For a Hilbert space H, given a probability measure* $\mu : \mathscr{P}(H) \mapsto \mathbb{R}^+$ *satisfying the conditions:*

- $0 \leq \mu(\pi) \leq 1$,*whenever* $\pi \leq \mathbb{I}_H$
- $\mu(\pi_1 + \pi_2 + \cdots) = \mu(\pi_1) + \mu(\pi_2) + \cdots$,*whenever* $\pi_1 + \pi_2 + \cdots \leq \mathbb{I}_H$
- $\mu(\mathbb{I}_H) = 1$

$\exists$ *a density operator* $\rho_\mu \in \mathscr{D}(H)$ *such that* $\mu(\pi) = \mathrm{Tr}[\pi \rho_\mu]$.

In other words, the only probability measure that one can assign to the set of positive operators for a given Hilbert space is through the Born rule on density operators. The formal proof is quite extensive so it is not provided in the thesis.

2.4.2 Process Matrices

Motivated from Gleason's theorem suppose we wish to assign a probability measure on all possible experiments that two parties Alice and Bob wish to do in the Local Laboratories. Here we won't make any assumption regarding there being some causal order between the actions of Alice and Bob. Furthermore, we assume that each party open's their lab only one's to get some information from the environment, then performs an experiment and sends the updated state back into the environment. According to Quantum theory the most general Experiment that Alice can perform is given by a Quantum Instrument $\mathbf{I}_A = \{\Lambda_A^a | \Lambda_A^a : \mathscr{B}(H_{A_I}) \mapsto \mathscr{B}(H_{A_O})\}$. The condition that this indeed is a valid quantum instrument is represented in terms of the CJ operators [2.1.7] as $J_{\Lambda_A^a} \geq 0$ and $\mathrm{Tr}_{A_O}[\sum_a J_{\Lambda_A^a}] = \mathbb{I}_{A_I}$. Here the index a denotes the classical outcome of the Quantum Instrument and H_{A_I} and H_{A_O} denote the associated Hilbert spaces for the input and output quantum system of Alice. Similarly one can also define Bob's most general experiment as a Quantum instrument $\mathbf{I}_B = \{\Lambda_B^b | \Lambda_B^b : \mathscr{B}(H_{B_I}) \mapsto \mathscr{B}(H_{B_O})\}$ obeying similar conditions. What we wish to do is to prescribe a probability measure for all possible outcomes of all possible instruments of Alice and Bob. To do this it is convenient to describe all possible Instrument Outcomes of the parties not as abstract maps but their associated CJ operators. Let us define the sets

$$\mathbf{M}_A := \{M_{A_I A_O} | M_{A_I A_O} \geq 0\}, \tag{2.62a}$$

$$\mathbf{M}_B := \{M_{B_I B_O} | M_{B_I B_O} \geq 0\}, \tag{2.62b}$$

The above sets basically denote the CJ operators of all CP maps for the respective parties. Without assuming any background causal structure between Alice's and Bob's actions, the most general statistics is given by a bi-linear functional P satisfying

$$P : \mathbf{M}_A \times \mathbf{M}_B \mapsto [0, \infty),\ s.t. \tag{2.63a}$$

$$P(\mathbb{M}_{A_I A_O}, \mathbb{M}_{B_I B_O}) = 1,\ \forall\ \mathbb{M}_{A_I A_O}, \mathbb{M}_{B_I B_O} \tag{2.63b}$$

$$\text{satisfying } \mathrm{Tr}_{A_O}[\mathbb{M}_{A_I A_O}] = \mathbb{I}_{A_I}, \mathrm{Tr}_{B_O}[\mathbb{M}_{B_I B_O}] = \mathbb{I}_{B_I}.$$

The above conditions play the role of the constraints mentioned in Theorem [5]. The bold letter symbol $\mathbb{M}_{A_I A_O}$ is used to denote the CJ operator of a CPTP map. The equation Eq.(2.63a) ensures positivity of probabilities and Eq.(2.63b) ensures that the sum of the probabilities of all outcome of Alice and Bob is unity. Using an unentangled version of Gleason-Busch theorem [57], any such bi-linear functional can be written as

$$P\left(M_{A_I A_O}, M_{B_I B_O}\right) = \mathrm{Tr}\left[W_{A_I A_O B_I B_O}\left(M_{A_I A_O} \otimes M_{B_I B_O}\right)\right], \tag{2.64}$$

where $W_{A_I A_O B_I B_O} \in \mathrm{Herm}(H_{A_I} \otimes H_{A_O} \otimes H_{B_I} \otimes H_{B_O})$ is a Hermitian operator. The requirement (2.63a) ensures $W_{A_I A_O B_I B_O}$ to be a positive-on-product-test (POPT) operator. Furthermore, the requirement that Alice and Bob get valid statistics even when supplemented with arbitrary entangled states gives us

$$\left\{ \begin{array}{c} \mathrm{Tr}[(\rho_{A'_I B'_I} \otimes W)(M_{A'_I A_I A_O} \otimes M_{B'_I B_I B_O})] \geq 0, \\ \forall\ M_{A'_I A_I A_O} \geq 0,\ M_{B'_I B_I B_O} \geq 0,\ \rho_{A'_I B'_I} \geq 0 \end{array} \right\}, \tag{2.65}$$

Eq.(2.65) ensures $W_{A_I A_O B_I B_O}$ to be a positive operator, *i.e.*, $W_{A_I A_O B_I B_O} \geq 0$. Such a positive operator satisfying the normalization condition (2.63b) is called a process matrix [11]. Adopting a notational convention followed in [58]:

$$_S M_{RS} := \frac{1}{d_S}\left(\mathrm{Tr}_S[M_{RS}]\right) \otimes \mathbb{I}_S, \tag{2.66}$$

where, d_S denotes the dimension of the Hilbert space H_S. Mathematically, the normalization condition can be conveniently written using this notation as

$$\left\{\begin{array}{c} {}_{A_IA_OB_IB_O}W = \frac{1}{d_{A_I}d_{B_I}}\mathbb{I}_{A_IA_OB_IB_O}, \\ {}_{A_IA_O}W = {}_{A_IA_OB_O}W, \ {}_{B_IB_O}W = {}_{B_IB_OA_O}W, \\ W = {}_{A_O}W + {}_{B_O}W - {}_{A_OB_O}W \end{array}\right\}. \tag{2.67}$$

Often we will avoid the suffixes of Hilbert spaces to avoid cluttering of notation.

2.4.3 Causally Separable and Causally nonseparable Processes

The set of process matrices can be of two types: (i) causally separable and (ii) causally nonseparable. Here we formally define the notion of causally (non)separable processes for bipartite case. These notions can also be generalized for the multipartite case as well.

Definition 27. ***Causally Separable Process:*** *A bipartite process W^{CS} is called causally separable if it admits the following convex decomposition*

$$W^{CS} := p_1 W^{A\prec B} + p_2 W^{B\prec A} + p_3 W^{B\not\prec\not\succ A}, \tag{2.68}$$

where $W^{A\prec B}$ ($W^{B\prec A}$) denotes a process where Alice (Bob) is in the causal past of Bob (Alice), $W^{B\not\prec\not\succ A}$ represents a process where Alice's and Bob's actions are spacelike separated, and $\vec{p} = (p_1, p_2, p_3)$ denotes a probability vector.

A process is called ***causally nonseparable*** if it is not causally separable. Mathematically causally separable processes $W^{A\prec B}, W^{B\prec A}$ and $W^{B\not\prec\not\succ A}$ satisfy [58]

$$\left\{\begin{array}{c} W^{A\prec B} = {}_{B_O}W^{A\prec B}, \ {}_{B_IB_O}W^{A\prec B} = {}_{A_OB_IB_O}W^{A\prec B}, \\ W^{B\prec A} = {}_{A_O}W^{B\prec A}, \ {}_{A_IA_O}W^{B\prec A} = {}_{B_OA_IA_O}W^{B\prec A}, \\ W^{B\not\prec\not\succ A} = {}_{A_OB_O}W^{B\not\prec\not\succ A} \end{array}\right\}. \tag{2.69}$$

Alternatively, a causally separable process can also be expressed as

$$W^{CS} := pW^{A\not\prec B} + (1-p)W^{B\not\prec A}, \tag{2.70}$$

for $p \in [0,1]$, where $W^{A\not\prec B}$ ($W^{B\not\prec A}$) denotes a process where communication from Alice (Bob) to Bob (Alice) is impossible. The authors in [11], first reported an example of an bipartite nonseparable process. The causal indefiniteness is established through a causal inequality which is discussed in more detail in the following subsection.

2.4.4 Causal Inequalities

The authors in [11] derived a causal inequality under four assumptions:

- **Definite causal structure:** The actions of Alice and Bob are embedded in a definite spacetime structure. In other words their actions are either timelike or spacelike seperated or they are in convex mixture of such definite structures.
- **Free choice:** Each party can freely generate random classical variables say x for Alice and y for Bob. This assumption is the same as that assumed by Bell while deriving Bell inequalities.
- **Closed laboratories:** Alice's outcome a can be correlated with Bob's variable y only if the latter is generated in the causal past of the system entering Alice's laboratory. Analogously, b can be correlated with a x only if the latter is generated in the causal past of the system entering Bob's laboratory.
- **Time order in local labs:** Locally both the Laboratories are assumed to have a definite order of time. Meaning some external system enters Alice's Lab ,then she performs a Quantum Instrument which may or may not be a function of x and them she sends the evolved state out of the lab after getting the outcome a. Similarly Bob's Lab is also assumed to have this ordering.

Violation of the causal inequality with the last three assumptions holding true, establishes indefiniteness of causal structure. Subsequently, a symmetric variant of the causal inequality or causal game is studied by [59] namely the guess your neighbour's input (GYNI) game.

Guess Your Neighbour's Input (GYNI) game

The simplest version of the game involves two distant players, Alice and Bob. Alice (Bob) tosses a random coin to generate a random bit x $(y) \in \{0,1\}$. Each party aims to guess the coin state of the other party. Denoting their guesses as a

and b respectively, the success probability reads as

$$P_{succ}^{\text{GYNI}} = \sum_{x,y=0}^{1} \frac{1}{4} P(a=y, b=x|x,y) \tag{2.71}$$

As it turns out for any causally separable process the success probability of GYNI is bounded by $1/2$ [59], leading to the causal inequality

$$P_{succ}^{\text{GYNI}} \leq 1/2. \tag{2.72}$$

Interestingly, their exist process matrices that lead to violation of this inequality, and thus establishes causal indefiniteness. An explicit such example was provided by Cyril and other's in [59]. Consider the process and quantum instruments:

$$\begin{Bmatrix} W_{A_I A_O B_I B_O}^{Cyril} = \frac{1}{4}\left[\mathbb{I}^{\otimes 4} + \frac{1}{\sqrt{2}}\left(\sigma^3\sigma^3\sigma^3\mathbb{I} + \sigma^3\mathbb{I}\sigma^1\sigma^1\right)\right], \\ \mathbf{I}_X^{(0)} \equiv \left\{M_{X_I X_O}^{0|0} = 0,\ M_{X_I X_O}^{1|0} = 2\left|\phi^+\right\rangle\left\langle\phi^+\right|\right\}, \\ \mathbf{I}_X^{(1)} \equiv \left\{M_{X_I X_O}^{0|1} = (|0\rangle\langle 0|)^{\otimes 2},\ M_{X_I X_O}^{1|1} = (|1\rangle\langle 1|)^{\otimes 2}\right\}, \end{Bmatrix}, \tag{2.73}$$

where $X \in \{A, B\}$.(In $M_{X_I X_O}^{r|s}$, s denotes various choice of instruments and r represent outcomes of those instrument). The tensor product symbol is sometimes omitted to avoid cluttered notation. The above protocol gives us the following correlation between Alice and Bob:

$$p^{Cyril}(ab|xy) = \text{Tr}[W_{A_I A_O B_I B_O}^{Cyril}(M_{A_I A_O}^{a|x} \otimes M_{B_I B_O}^{b|y})] \tag{2.74}$$

If we substitute Eq.(2.74) in Eq.(2.71) we yield a success $P_{succ}^{\text{GYNI}} = 5/16(1+1/\sqrt{2}) \approx 0.5335 > 1/2$. Numerical evidence suggests possibility of other quantum processes leading to higher success [59]. At this point one may ask the following question: *Is it always possible to demonstrate the causal nonseparability of a quantum process via violation of a causal inequality?* Feix, Araujo, and Brukner [60] have shown the existence of a bipartite causally nonseparable process which never the less only yields causal correlations (see also [61]). Such class of processes are termed as causal processes.

2.4.5 Causal and Extensibly causal Processes

Definition 28. ***Causal Process:*** *A bipartite process W^C is called causal if any correlation obtained from such a process can always be written as convex decomposition of one way no-signalling correlations.*

Denoting $\{M^{a|x}_{A_IA_O}\}$ and $\{M^{b|y}_{B_IB_O}\}$ as arbitrary choice of instruments for Alice and Bob any correlation obtainable from a causal process W^C and arbitrary instruments $\{M^{a|x}_{A_IA_O}\}$ and $\{M^{b|y}_{B_IB_O}\}$ can be decomposed as

$$\mathrm{Tr}[W^C(M^{a|x}_{A_IA_O} \otimes M^{b|y}_{B_IB_O})] := p(ab|xy) = p \times p^{A \prec B}(ab|xy) + (1-p) \times p^{B \prec A}(ab|xy) \quad (2.75)$$

Where $p \in [0,1]$ and $p^{A \prec B}(ab|xy)$ and $p^{B \prec A}(ab|xy)$ satisfy

$$\sum_b p^{A \prec B}(ab|xy) = \sum_b p^{A \prec B}(ab|xy') \; \forall a,x,y,y' \quad (2.76a)$$

$$\sum_a p^{A \prec B}(ab|xy) = \sum_a p^{A \prec B}(ab|x'y) \; \forall b,x,x',y \quad (2.76b)$$

The existence of such process shows that some causally nonseparable processes can never be tested in a device-independent manner (meaning they would never violate a causal inequality). Having said this, its natural to ask if causal processes remain causal when assisted with some entangled state between the parties. Oreshkov and Giarmatzi in [61] show an explicit example of a tripartite process that is causal but becomes noncausal if an additional entangled state is provided. This motives the authors in [60] and [61] to come up with the notion of extensibly causal processes.

Definition 29. ***Extensibly Causal Process:*** *A bipartite process W^{EC} is called extensibly causal if $W^{EC} \otimes \rho_{AB}$ is causal for any entangled state ρ_{AB}.*

At this point its not clear whether there are causally nonseparable processes which are extensibly causal? [60] provide numerical evidence for the existence of such extensibly causal processes. If we denote $\mathbf{W}, \mathbf{W^C}, \mathbf{W^{EC}}$ and $\mathbf{W^{CS}}$ as the set of all processes, set of all causal processes, set of all extensibly causal processes and set of all causally Separable processes respectively then we have the following subset realtion:

$$\mathbf{W^{CS}} \subseteq \mathbf{W^{EC}} \subsetneq \mathbf{W^C} \subsetneq \mathbf{W} \quad (2.77)$$

Where we have numerical evidence for the strict subset relation $\mathbf{W^{CS}} \subsetneq \mathbf{W^{EC}}$.

Before moving on to the next subsection here we point out the notion of Genuine Indefiniteness for Multipartite Processes. For instance in the tripartite scenario one can define the notion of bi-causal processes as following

Definition 30. ***Bicausal Process:*** *A tripartite quantum process W is called bi-causal if it allows a convex decomposition $W = \sum_i p_i W_i$, where each W_i are causally separable across some bi partition.*

Processes that are not bi-causal are termed as ***genuinely causally non-separable***. Recalling Eq.(2.70), a tripartite bi-causal process W can always be written as

$$\begin{aligned} W =& p_1 W^{A \not\prec BC} + p_2 W^{B \not\prec AC} + p_3 W^{C \not\prec AB} \\ &+ p_4 W^{BC \not\prec A} + p_5 W^{AC \not\prec B} + p_6 W^{AB \not\prec C}, \end{aligned} \tag{2.78}$$

with $\{p_i\}_{i=1}^{6}$ denoting a probability vector. Here, the term $W^{A \not\prec BC}$ denotes a process where Alice cannot communicate to Bob or Charlie, whereas in process $W^{BC \not\prec A}$ neither Bob nor Charlie can communicate to Alice. The other terms carry similar meanings. Importantly, in the process $W^{A \not\prec BC}/W^{BC \not\prec A}$ causal inseparability could be present between Bob and Charlie.

Chapter 3

Role of System Composition in Timelike Correlations

3.1 Introduction

In quantum theory, the postulates prescribe a specific composition rule for systems consisting of multiple subsystems. When individual subsystems are considered quantum, various mathematically consistent models can be constructed to describe the state and effect spaces of multipartite systems (see Section 2.2.5). The extreme cases of composition are represented by the minimal and maximal tensor product constructions, denoted as *SEP* and $\overline{SEP}$, respectively. Intermediate between these two limits, alternative composition rules can be formulated, among which the quantum composition (**Q**) is a notable example.

A natural question arises: do there exist input-output correlations that are characteristic of specific composite structures? A compelling answer is provided by Bell nonlocal correlations [1, 62–64], which are absent in the *SEP* theory but present in all other compositions. Conversely, a *no-go* result follows from the work of Barnum *et al.* [6], demonstrating that any bipartite spacelike correlation achievable in $\overline{SEP}$ is also attainable in **Q**. Indeed, any composite model of two quantum systems that adheres to the no-signaling principle and local tomography cannot exhibit correlations that surpass quantum limits in spacelike-separated scenarios (See section [2.3]).

One might conjecture that any input-output correlation attainable within *SEP* should also be achievable within **Q**, given the interchangeability of state and effect cones in *SEP* and $\overline{SEP}$ theories (see Fig. [3.1]. However, in this chapter, we demonstrate that such an intuition is incorrect. Specifically, there exist timelike correlations in *SEP* that cannot be realized within the quantum composition model **Q**. Moreover, the existence and strength of such correlations

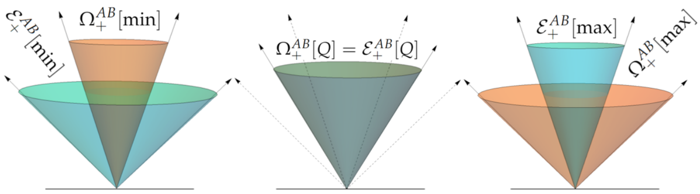

FIG. 3.1 Different possible compositions of two elementary quantum systems. Left: minimal tensor product composition – allows only separable states but effect cone is enlarged. Right: maximal tensor product composition – allows only separable effect but state cone is enlarged. Middle: quantum composition, state and effect cones are identical (self dual).

may vary across different composite models, thereby providing empirical means to distinguish among them. In other words, bipartite compositions of elementary quantum systems can exhibit beyond-quantum timelike correlations, even when the individual subsystems adhere to quantum mechanics.

To substantiate this claim, we propose a communication task (game) involving two parties. We first analyze the optimal qubit communication required to accomplish the task under the assumption of quantum composition. A necessary and sufficient condition for the perfect execution of the task is derived within the generalized probabilistic theory (GPT) framework. Subsequently, we establish that, under the *SEP* composition, the required number of communicated qubits for perfect task completion is strictly lower than in the quantum composition case. Consequently, *SEP* composition trivializes certain communication complexity problems that remain nontrivial under quantum composition.

It is noteworthy that the information-theoretic axiom *"communication complexity is nontrivial"* has been instrumental in identifying unphysical consequences of beyond-quantum spacelike correlations [7, 8, 65–68]. Our study demonstrates that this axiom is equally effective in isolating beyond-quantum correlations within the timelike domain. The proposed communication task thus provides an empirically testable criterion for the natural selection of bipartite composite structures among the various possible compositions that lie between *SEP* and $\overline{SEP}$.

3.2 Perfect Distinguishability of non orthogonal quantum states in SEP theory.

In this section, we present a significant result demonstrated by [69], which establishes that non-orthogonal quantum states can, under certain conditions, be perfectly distinguishable within the framework of separable (*SEP*) theory. Consider the following product states:

$$|\psi_1\rangle = |0\rangle \otimes |0\rangle \tag{3.1a}$$

$$|\psi_2\rangle = |+\rangle \otimes |+\rangle . \tag{3.1b}$$

It is evident that these two states cannot be perfectly distinguished within standard quantum theory (see Theorem [1]). Given that $|\psi_1\rangle$ and $|\psi_2\rangle$ are valid states under the *SEP* composition of two qubits, it is natural to investigate whether they can be distinguished using measurements permitted by *SEP* theory. To address this question, consider the following measurement operators:

$$E_1 = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 1 \end{pmatrix}. \tag{3.2}$$

Now one can show that $E_1, E_2 \in \mathscr{E}^{SEP}$, indicating that these operators are indeed permissible measurement effects within *SEP* theory. The detailed argument is provided in Appendix [A.1] as part of a proof. Moreover, note that $E_1 + E_2 = \mathbb{I}$ and thus they constitute a valid measurement in *SEP* theory.

Furthermore, it can be readily verified that

$$\mathrm{Tr}[E_i |\psi_j\rangle \langle\psi_j|] = \delta_{ij}, \quad \forall i, j \in \{1, 2\}, \tag{3.3}$$

which implies perfect distinguishability of $|\psi_1\rangle$ and $|\psi_2\rangle$ within the *SEP* theory. This result reveals a striking and counterintuitive consequence of *SEP* composition, wherein non-orthogonal quantum states become perfectly distinguishable. In the subsequent section, we introduce the Pairwise Distinguishability Game to further explore the implications of such composition rules in information-theoretic tasks.

3.3 Pairwise distinguishability game ($\mathscr{P}_D^{[n]}$)

The *Pairwise Distinguishability Game*, denoted by $\mathscr{P}_D^{[n]}$, involves two players, Alice and Bob, along with a Referee. In each iteration of the game, the Referee

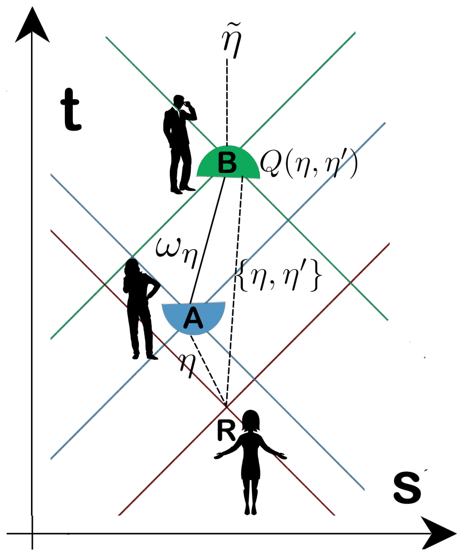

FIG. 3.2 Pairwise Distinguishability Game

provides Alice with a classical message η, randomly selected from a finite set of messages $\mathbf{N}$. The superscript n in $\mathscr{P}_D^{[n]}$ represents the cardinality of the set $\mathbf{N}$, i.e., $n = |\mathbf{N}|$. Subsequently, the Referee presents Bob with a query regarding Alice's message: whether the message given to Alice in that round was η or some alternative $\eta' \in \mathbf{N}$, where $\eta' \neq \eta$. This query is denoted by $\mathbb{Q}(\eta,\eta')$ and most importantly is unknown to Alice. Since $\eta' \neq \eta$, there exist $\binom{n}{2}$ distinct queries of this form. The objective of the game is for Bob to answer all such queries correctly (see Fig. [3.2]). Notably, Alice and Bob do not share any pre-existing correlation; however, Alice is permitted to encode her message onto the states of a physical system and transmit these states to Bob. The following proposition establishes a necessary and sufficient condition for achieving a perfect winning strategy in any Generalized Probabilistic Theory (GPT).

Proposition 1. *A perfect winning strategy for the game* $\mathscr{P}_D^{[n]}$ *necessitates that Alice encodes her message* $\eta \in \mathbf{N}$ *into a set of states* $\{\omega_\eta\}_{\eta\in\mathbf{N}} \subset \Omega$ *of some system* $S \equiv (\Omega, E)$ *such that the states in* $\{\omega_\eta\}_{\eta\in\mathbf{N}}$ *are pairwise distinguishable.*

Proof. Suppose Alice employs a set of states $\{\omega_\eta\}_{\eta\in\mathbf{N}}$ that are not pairwise distinguishable. Then, there exist at least two states $\omega_\alpha, \omega_\beta \in \{\omega_\eta\}_{\eta\in\mathbf{N}}$ that are not distinguishable. If the Referee asks Bob the query $\mathbb{Q}(\alpha,\beta)$, he will necessarily provide an incorrect answer with some nonzero probability due to the indistinguishability of ω_α and ω_β. Consequently, for a perfect winning strategy in $\mathscr{P}_D^{[n]}$, the information dimension $\mathbf{I}_D(S)$ of the system S must satisfy the lower bound $\mathbf{I}_D(S) \geq n$. □

Next we show that if we consider two qubit communication from Alice to Bob to play the game then *SEP* composition between the two qubits performs better than the ordinary Quantum Composition. We also show that the advantage can also be scaled arbitrarily with increasing value of n.

3.4 Characterizing Strong Timelike Correlations of SEP Composition in $\mathscr{P}_D^{[n]}$

Our next result demonstrates that if we consider the $\mathscr{P}_D^{[12]}$ game then two qubits in *SEP* composition can perfectly win the game while 4 qubits are required if the composition is quantum.

Theorem 6. *Four qubits communication from Alice to Bob is required for winning the game $\mathscr{P}_D^{[12]}$ when quantum composition is considered among the elementary systems, whereas two SEP-bits (i.e. two qubits in SEP composition) suffice for winning this game.*

Proof. (outline) As argued in [2.2.3] for a quantum system, the information dimension is the same as its operational dimension, which is again the same as the dimension of the associated Hilbert space. Since the Hilbert space dimension of $(\mathbb{C}^2)^{\otimes 3}$ is 8, according to Proposition [1], three qubits communication is not sufficient for winning the game $\mathscr{P}_D^{[12]}$ perfectly. However, four qubits communication suffices as the number of distinguishable (as well as pairwise distinguishable) states, in this case, are 16. If we consider the *SEP* composition between two qubits, then the following 12 states $\mathbf{A} := \{|\kappa\kappa\rangle, |\kappa\bar{\kappa}\rangle, |\bar{\kappa}\kappa\rangle, |\bar{\kappa}\bar{\kappa}\rangle\}_{\kappa\in\{x,y,z\}}$ turn out to be pairwise distinguishable; where $|\alpha\beta\rangle := |\alpha\rangle \otimes |\beta\rangle$ and $|\kappa\rangle$ ($|\bar{\kappa}\rangle$) is the eigenstate of Pauli operator σ_κ with eigenvalue $+1$ (-1), where $\kappa \in \{x,y,z\}$. While some pair of states, such as $\{|xx\rangle, |x\bar{x}\rangle\}$, are perfectly distinguishable in quantum theory due to mutual orthogonality, some pairs, such as $\{|xx\rangle, |zz\rangle\}$, consisting non-orthogonal states cannot be perfectly distinguished in quantum theory. However, as shown in Section [3.2] such states are distinguishable in *SEP* theory. Complete analysis (along with the measurement) of pairwise distinguishability of the states $\mathbf{A}$ in *SEP* theory is presented in Appendix [A.1]. The game $\mathscr{P}_D^{[12]}$, thus, can be perfectly won just by using two qubits in *SEP* composition. This completes the proof of our claim. □

The above theorem leads to an interesting observation

Corollary 1. *SEP composition exhibits the phenomenon of dimension mismatch.*

Proof. Operational dimension of two SEP-bits is 4, which follows from Proposition 2.5 of [69], whereas our Theorem [6] establishes its information dimension to be strictly greater than 4, and completes the proof. □

Dimension mismatch and consequently the presence of stronger than quantum timelike correlation in the above result arises strictly from the choice of composition. One might ask whether the advantage of two qubits in *SEP* composition for playing the $\mathscr{P}_D^{[n]}$ game can be made arbitrarily large. Our next result is a no-go answer to this question.

Lemma 1. *The game $\mathscr{P}_D^{[n]}$ cannot be won perfectly by communicating the encoded states chosen from the SEP composition of two qubit system ($\mathbb{C}^2 \otimes_{\min} \mathbb{C}^2$) whenever $n > 12$.*

Proof. System $S \equiv (\Omega, E)$, having the information dimension $\mathbf{I}_D(S) = k$ can always be used to encode the classical message to win the game $\mathscr{P}_D^{[n\leq k]}$ perfectly. As per definition, $\mathbf{I}_D(S) = k$ provides them achievable upper bound on the number of pairwise distinguishable states. Here we will calculate the Information dimension of the most elementary SEP i.e., $\mathbb{C}^2 \otimes_{\min} \mathbb{C}^2$, and show that $\mathbf{I}_D(\mathbb{C}^2 \otimes_{\min} \mathbb{C}^2) = 12$. So no $\mathscr{P}_D^{[n>12]}$ game can be won perfectly considering the above system. As shown by [69], two pure states $\rho_1 = \rho_1^A \otimes \rho_1^B$ and $\rho_2 = \rho_2^A \otimes \rho_2^B$ are perfectly distinguishable in SEP if and only if

$$\mathrm{Tr}\,\rho_1^A \rho_2^A + \mathrm{Tr}\,\rho_1^B \rho_2^B \leq 1. \tag{3.4}$$

Expressing $\rho_i^X = \frac{1}{2}\left(\mathbf{1} + \hat{n}_i^X \cdot \vec{\sigma}\right)$, where $\hat{n}_i^X \in \mathbb{R}^3$ with $|\hat{n}_i^X| = 1$ for $i \in \{1,2\}$ and for $X \in \{A, B\}$, the above condition can be re-written as

$$\hat{n}_1^A \cdot \hat{n}_2{}^A + \hat{n}_1^B \cdot \hat{n}_2^B \leq 0. \tag{3.5}$$

Let us define $N_i := (\hat{n}_i^A, \hat{n}_i^B)^T \in \mathbb{R}^3 \oplus \mathbb{R}^3 \equiv \mathbb{R}^6$. Thus the above condition reads as

$$N_1^T . N_2 \leq 0. \tag{3.6}$$

Therefore, our question boils down to finding the maximum number of vectors in $\mathbb{R}^6$ such that the inner product between any two vectors is not positive definite.

It is not hard to argue that in a d-dimensional vector space at the most $2d$ number of such vectors can be drawn. To appreciate the argument, observe that, in 1-D only two such vectors can be drawn trivially, while in 2-D the number is *four*. Now, for $\mathbb{R}^d$ we can decompose it as $\mathbb{R}^{d-1} \oplus \mathbb{R}$. As mentioned earlier the $d^{th}-$dimension $\mathbb{R}$ contains only two such vectors and $\mathbb{R}^{d-1}$ can be further decomposed as $\mathbb{R}^{d-2} \oplus \mathbb{R}$. Thus, it can be argued from the method of induction that the maximum number of such vectors is $2d$. □

However the advantage of *SEP* composition over quantum theory can be increased if we start with more number of *SEP*-bits initially.

Theorem 7. *$2k$ number of SEP-bits are sufficient for winning the game $\mathscr{P}_D^{[12^k]}$ perfectly, whereas it requires $2k + \lceil k \log_2 3 \rceil$ number of qubits, with $k \in \mathbb{Z}_+$.*

Proof is provided in Appendix [A.2]. Now we move to a possible experimental implication of our study. Novel experimental proposals bring adequate physical reasoning to the 'mathematical fiction' of Hilbert space formulation of quantum theory. For instance, experimentally observed algebraic relationship among the coherent cross sections of scattering amplitudes in triple-slit experiment constitutes a test for complex versus quaternion quantum theory [70]. The experiment by Sinha *et al.* that rules out multi-order interference in quantum mechanics is worth mentioning at this point [71]. In a similar spirit, a pertinent question to ask is which particular composite structure between two elementary qubits must be preferred [72]. At this point, one might wish to postulate a particular composite structure. Schrodinger, for instance, found quantum composition "rather discomforting" [73] due to the peculiarity of quantum entanglement as demonstrated in the Einstein-Podolsky-Rosen gedanken experiment [47]. The SEP composition, which does not contain these "discomforting" features, is immediately ruled out due to the seminal experiment by Aspect and collaborators [3] and the recent *loophole-free* Bell tests [74–76] which validate the presence of nonlocal correlations in nature. There are still a number of different bipartite compositions, such as SEP+δ_{EP}, $\overline{\text{SEP}} - \xi_{EP}$, and $\overline{\text{SEP}}$, that, like the quantum composition **Q**, incorporate nonlocal correlations and, hence, cannot be excluded from the Bell test's results. Furthermore, no such model can contain any spacelike correlation which is not available in quantum theory [6]. At this point, our $\mathscr{P}_D^{[12]}$ game starts playing a crucial role. Perfect success of this game with communication of less than four qubits assures the presence of beyond quantum timelike correlation which indicates a departure from the composition rule adopted in quantum theory. In fact, our next result proposes a generic test in this direction.

Proposition 2. *For $n > 2^k$, the game $\mathscr{P}_D^{[n]}$ cannot be won with k qubits communication from Alice to Bob if the composition rule is quantum.*

The proof simply follows from Proposition [1] and the fact that the information dimension of a quantum system is the same as its Hilbert space dimension. A non-null result in Proposition [2], *i.e.*, successful completion of the task $\mathscr{P}_D^{[>2^k]}$ with k qubits communication, will indicate a departure from the quantum composition rule, whereas null result builds confidence toward quantum composition.

3.5 Discussion

The present work introduces a novel paradigm for experimentally testing the derivation of the composition rule within quantum mechanics. Notably, unlike the seminal Bell tests, which investigate spacelike correlations, our approach is based on timelike correlations. In this context, recent contributions in Refs. [46, 77] are particularly relevant. Specifically, Ref. [46] presents instances of timelike correlations that exceed those predicted by quantum mechanics, considering systems composed of elementary units. However, the elementary systems analyzed therein exhibit postquantum characteristics, particularly the so-called *square bits*, which inherently generate stronger timelike correlations than a qubit, as demonstrated through our $\mathscr{P}_D^{[n]}$ task. In contrast, the stronger-than-quantum timelike correlations observed in the present study emerge strictly due to the choice of composition between qubits.

From a technical standpoint, Ref. [46] employs the concept of *signaling dimension*, originally introduced in Ref. [45], whereas our analysis relies crucially on the notion of *information dimension*. Additionally, the approach in Ref. [77] involves the computation of entropic quantities, which necessitate a well-defined theoretical framework [78, 79]. In contrast, our methodology is founded on the intuitive principle of pairwise distinguishability.

Furthermore, our findings give rise to several intriguing questions that warrant further investigation. Notably, the dimension mismatch explored in Ref. [64] for square bit models suggests various nontrivial consequences, such as the potential collapse of communication complexity and implications for Maxwell's demon, possibly violating the second law of thermodynamics. Similar analyses in our context are particularly interesting, given that Corollary [1] implies a dimension mismatch that arises strictly from the compositional structure of local quantum systems. From a complexity theory perspective, addressing the question posed following Theorem [7] is of significant interest. Additionally, exploring the implications of alternative composition rules for higher-dimensional elementary quantum systems remains an open direction. Such inquiries could also be reformulated within the framework of field-theoretic formalism [80–83].

Moreover, investigating the role of stronger timelike correlations within the generalized probabilistic theory framework may yield profound insights into the fundamental structure of spacetime. The recently studied polygonal models [34–37, 40] could serve as an initial step toward such an exploration.

Chapter 4

Foundational Implications of Nonlocal behavior in Composite Polygon Models

4.1 Introduction

Entanglement, a fundamental feature of composite quantum systems, marks a key departure from classical physics [84]. While maximally entangled states yield completely mixed marginals, non-maximally entangled states exhibit partially mixed marginal states. Under local operations and classical communication (LOCC), the former are generally more useful. However, in alternative paradigms, non-maximal entanglement can be advantageous, as evidenced in nonlocality studies [1, 62–64]. For instance, while the Clauser-Horne-Shimony-Holt (CHSH) inequality is maximally violated by maximally entangled states [4, 48], tilted CHSH inequalities favor non-maximally entangled states [85, 86]. Similarly, Hardy's test identifies nonlocality in any non-maximally entangled pure two-qubit state, whereas the maximally entangled state fails the test [49, 87, 88]. Such advantages extend to Bayesian games [89] and reverse zero-error channel coding [90].

A crucial question is whether non-maximal entanglement is a uniquely quantum phenomenon. To explore this, we consider generalized probability theories (GPTs) (see Section [2.2]) [34–36, 64, 91, 92]. While simple toy models, such as the square-state system, permit stronger-than-quantum nonlocality [7], they lack direct analogues of non-maximally entangled states, suggesting a quantum-specific feature. However, we demonstrate that this intuition is incorrect, leading to significant new insights.

We examine GPTs with state spaces described by regular polygons [34] and analyze bipartite entangled structures. While prior work focused on maximally entangled states in these models [34], we identify additional entanglement classes. Specifically, bipartite pentagon systems yield two distinct entanglement classes, whereas hexagon systems admit six. Examining their nonlocal properties, we find that, unlike even-gon models that exhibit supra-Tsirelson CHSH violations [34], odd-gon models remain within Tsirelson's bound but reveal beyond-quantum Hardy-type nonlocality [49] (see Section [2.3.1]). Furthermore, certain mixed entangled states in these models, unlike their quantum counterparts, exhibit Hardy-type nonlocality.

Lastly, we investigate post-quantum correlations, particularly *almost quantum correlations*, which respect all known bipartite physical principles [93]. Models reproducing these correlations typically violate the no-restriction hypothesis [94]. However, we identify a nontrivial subset of strictly almost quantum correlations emerging from bipartite pentagon models without violating this hypothesis, strengthening prior known results [34].

4.2 Entanglement classes in Composite Polygons

As previously discussed, the bipartite composition of any Generalized Probabilistic Theory (GPT) system must be confined between the minimal and maximal tensor product compositions (see Section [2.2.4]). For polygon systems, any such composition must encompass all product states and product effects. The sets of extremal states and ray extremal effects are defined as follows:

$$\mathscr{P}_{st}[n] \equiv \left\{\omega_{n(i-1)+j} := \omega_i^A \otimes \omega_j^B\right\}_{i,j=1}^{n}, \tag{4.1a}$$

$$\mathscr{P}_{ef}[n] \equiv \left\{e_{n(i-1)+j} := e_i^A \otimes e_j^B\right\}_{i,j=1}^{n}, \tag{4.1b}$$

where ω_i^A and ω_j^B are defined in Eq. (2.41), and e_i^A and e_j^B are defined in Eq. (2.44). Instead of representing a bipartite state or effect as a vector in $\mathbb{R}^9$, it is sometimes more convenient to express it as a 3×3 matrix, i.e., $\omega_i \otimes \omega_j \equiv \omega_i \otimes \omega_j^{\mathrm{T}}$.

Beyond the aforementioned product states, a bipartite composition may also admit non-factorizable states, commonly referred to as entangled states. For an n-gon system, one such entangled state was identified in [34]:

$$\Phi_J := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{for odd } n; \tag{4.2}$$

$$\Phi_J := \begin{pmatrix} \cos(\pi/n) & \sin(\pi/n) & 0 \\ -\sin(\pi/n) & \cos(\pi/n) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{for even } n. \tag{4.3}$$

As noted in [34], the state Φ_J serves as a natural analogue of the quantum mechanical maximally entangled state. The subscript follows the initials of the first author of [34]. It is crucial to ensure that any such entangled state yields consistent probabilities when measured with any product effect.

Similar to entangled states, a composite theory also allows for entangled effects, which must yield consistent probabilities when acting on any product state. In cases where a composite system accommodates both entangled states and entangled effects, care must be taken to ensure that all effects provide consistent probabilities across all states. For the composition of two square bits, it has been established that exactly four types of compositions exist [46]. Among these, one corresponds to the maximal composition, which permits all possible factorized and entangled states while admitting only factorized effects. Another extreme case is the minimal composition, which permits only factorized states but allows all possible factorized and entangled effects. The remaining two compositions lie strictly between these extremes.

In this work, we focus on the maximal composition of two polygon systems. For such a bipartite composition, the set of allowed reversible transformations is given by

$$\begin{aligned} \mathbb{T}_{AB}[n] := \Big\{ \mathbf{T}_{k_1}^{s_1} \otimes \mathbf{T}_{k_2}^{s_2},\ \text{Swap} \mid \mathbf{T}_{k_1}^{s_1} \in \mathbb{T}_n^A,\ \mathbf{T}_{k_2}^{s_2} \in \mathbb{T}_n^B \Big\}, \\ \text{where Swap}(\omega_i^A \otimes \omega_j^B) := \omega_j^A \otimes \omega_i^B, \\ s_1, s_2 \in \{\pm 1\},\ k_1, k_2 \in \{1, 2, \dots, n\}. \end{aligned} \tag{4.4}$$

It is important to note that the Swap operation, as well as local reversible transformations, map product states to product states and entangled states to entangled states. This property enables us to define equivalence classes of entangled states.

Definition 31. *Two entangled states Φ_1 and Φ_2 of a bipartite polygonal system are considered equivalent if they are connected by some local reversible transformation.*

For instance, in the maximal composition of two square bits, there exist precisely eight distinct entangled states [46]. However, all these states belong to the same equivalence class since they are connected via local reversible transformations. In contrast, as we will demonstrate, this is not necessarily the case for higher-order polygon systems.

4.2.1 Bipartite Pentagon system

Following the procedure mentioned in Appendix [B.1] in MATLAB, for bipartite composition of pentagon systems we obtain 135 different extreme states Φ_k; $k \in \{1,2,\cdots,135\}$. Among these we have 25 (say, number 1 to 25) factorized extreme states $\Phi_{5(i-1)+j} = \omega_i \otimes \omega_j^{\mathrm{T}}$, where $i,j \in \{1,\cdots,5\}$ and ω_i's are given in Eq.(2.41). The remaining 110 states (number 26 to 135) are entangled. Furthermore, applying the local reversible transformation we find that these states can be divided into two classes – (i) the first class contains 10 states (say, number 26 to 35), and (ii) the second one contains 100 states (number 36 to 135). One representation state of the first class is the state Φ_J of Eq.(4.2) and hence we call this *Janotta* class. One of the representation states for the second class is given by

$$\Phi_H = \begin{pmatrix} -\cos(\pi/5) & \frac{-r_5^6 \sin(\pi/5)}{8\left(1+r_5^2\right)} & 0 \\ \frac{-r_5^6 \sin(\pi/5)}{8\left(1+r_5^2\right)} & \cos(\pi/5) & \frac{-r_5^3}{4\sin(\pi/5)} \\ 0 & \frac{-r_5^3}{4\sin(\pi/5)} & 1 \end{pmatrix}. \tag{4.5}$$

We call this class the *Hardy* class, thus the sub-index 'H'. The justification of this nomenclature will become obvious in the next section. The other states in *Janotta* class and *Hardy* class can be obtained from the representative states Φ_J and Φ_H respectively by applying local reversible transformations.

The two classes of states Φ_J and Φ_H have structural distinctions. For instance the state in Eq.(4.2) we have $a_3 = a_6 = a_7 = a_8 = 0$. It can be shown that all the states in this class (obtained through local reversible transformations) have the same feature, which is not the case for the states in the class of Φ_H.

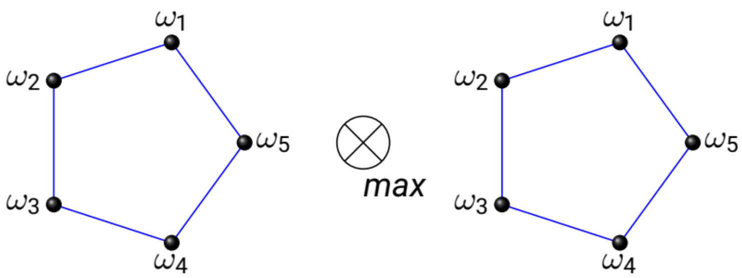

FIG. 4.1 Maximal tensor product of two elementary pentagon systems allows 25 product states. On the other hand, it allows two different classes (not equivalent under local reversible transformation) of entangled states: Janotta class states with Φ_J of Eq.(4.2) being a representative state and the Hardy class states with Φ_H of Eq.(4.5). While Φ_J can be thought of as a natural analog of the maximally entangled state of a two-qubit and does not exhibit Hardy's nonlocality, the Φ_H state shows Hardy and importantly with the success probability strictly greater than quantum success. However, the resulting correlation belongs to the set of *almost quantum set* Q_1.

4.2.2 Bipartite Hexagon system

The task of characterizing all the entangled states becomes computationally costly with higher gons. This is because the choices of eight different effects rapidly increase with the number of sides in the component polygons. However, we obtain a complete characterization of the entanglement states for the bipartite hexagon. It turns out that there are six different entangled classes of states possible there. Representation states for each of these classes are given below:

$$\Phi_I = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \Phi_{II} = \begin{pmatrix} \frac{1}{15r_6^2} & \frac{7}{10} & \frac{2}{3r_6^3} \\ \frac{7}{10} & \frac{1}{5r_6^2} & \frac{3}{5r_6} \\ \frac{2}{3r_6^3} & \frac{3}{5r_6} & 1 \end{pmatrix},$$
$$\Phi_{III} = \begin{pmatrix} \frac{1}{3r_6^2} & \frac{1}{r_6^4} & \frac{2}{3r_6^3} \\ \frac{1}{r_6^4} & 0 & \frac{1}{2r_6} \\ \frac{2}{3r_6^3} & \frac{1}{2r_6} & 1 \end{pmatrix}, \quad \Phi_{IV} = \begin{pmatrix} \frac{1}{7r_6^2} & \frac{9}{14} & \frac{10}{21r_6^3} \\ \frac{11}{14} & \frac{1}{7r_6^2} & \frac{5}{7r_6} \\ \frac{6}{7r_6^3} & \frac{3}{7r_6} & 1 \end{pmatrix}, \tag{4.6}$$
$$\Phi_{V} = \begin{pmatrix} \frac{1}{7r_6^2} & \frac{11}{14} & \frac{6}{7r_6^3} \\ \frac{9}{14} & \frac{1}{7r_6^2} & \frac{3}{7r_6} \\ \frac{10}{21r_6^3} & \frac{5}{7r_6} & 1 \end{pmatrix}, \quad \Phi_{VI} = \begin{pmatrix} \frac{-1}{2r_6^2} & \frac{-1}{r_6^4} & \frac{1}{3r_6^3} \\ \frac{-1}{r_6^4} & \frac{1}{2r_6^2} & \frac{-1}{2r_6} \\ \frac{1}{3r_6^3} & \frac{-1}{2r_6} & 1 \end{pmatrix}.$$

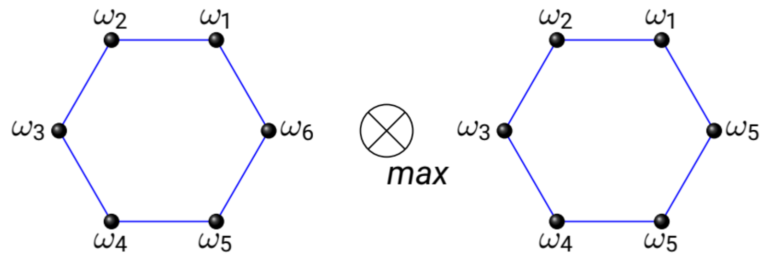

FIG. 4.2 Maximal tensor product of two elementary hexagon systems allows 36 product state. On the other hand, it allows six different classes of entangled states. Representative states for each of the classes are given in Eq.(4.6). The state Φ_I although can be thought of as an analog of the maximally entangled state, unlike the two-qubit maximally entangled state, exhibits Hardy's nonlocality behaviour.

The state Φ_I is equivalent to the maximally entangled state Φ_J of Eq. (4.3) as identified by Janotta *et al.* Important to note that the states Φ_I, Φ_{II}, Φ_{III} and Φ_{VI} are the symmetric representatives of their corresponding class, while we cannot find any symmetric state in class Φ_{IV} and Φ_V. Recall that, a joint state Φ^{AB} is called symmetric if $(e \otimes f)(\Phi^{AB}) = (f \otimes e)(\Phi^{AB})$, $\forall\ e, f \in V_+^\star$, and in matrix representation Φ^{AB} is symmetric if and only if the corresponding matrix is symmetric [34]. However, we can notice that the states Φ_{IV} and Φ_V are transposes of each other meaning that even though they are not connected by local reversible dynamics they are related by a swap operation. Thus the entanglement content in these two states is the same.

4.3 Insights on the nature of Quantum and Polygon model correlation

Quantum theory exhibits nonlocality; however, the precise extent to which quantum systems can demonstrate nonlocal behavior remains an open question. To gain a deeper understanding of the nature of quantum nonlocal correlations, it is essential to investigate the nonlocal properties of alternative theoretical models. In the following, we examine the Hardy nonlocal correlations [2.53] that can be realized within bipartite polygon models, with the aim of elucidating the fundamental characteristics of quantum nonlocal correlations.

Nonlocal properties of the correlations obtained from the maximally entangled states Φ_J have been studied in Ref. [34]. In particular, the maximal CHSH inequality violation has been explored for even and odd gons. Importantly, the correlations of even n systems can always reach or exceed Tsirelson's bound

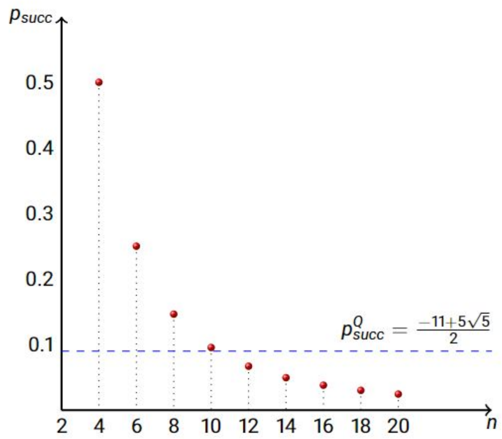

FIG. 4.3 (Color online) Red dots denote the maximum success probability of Hardy's nonlocality argument for maximally entangled states of bipartite even gons. The Blue dashed line denotes the optimal quantum success probability of Hardy's nonlocality argument which, in contrast, is obtained for the *non-maximally entangled state*.

$(2\sqrt{2})$, while the correlations of odd n systems are always below Tsirelson's bound. For the odd n systems the maximally entangled state belongs to the class of *inner product states*[1] and all correlations obtainable from measurements on inner product states satisfy Tsirelson's bound. Here we analyze the nonlocal properties of different classes of entangled states from the perspective of Hardy's nonlocality argument.

4.3.1 Hardy's nonlocality for maximally entangled states in polygon models

Here, we will analyze Hardy's nonlocality behaviour of the correlations obtained from the maximally entangled states of bipartite polygon theories. We prove two generic results. In the following, we first prove a no-go result.

Theorem 8. *The maximally entangled states Φ_J of the bipartite regular polygons with odd n do not exhibit Hardy's nonlocality argument.*

While maximally entangled states of bipartite odd gons do not exhibit Hardy's nonlocality, maximally entangled states of bipartite square bit do exhibit such nonlocality. The PR box correlation resulting from the maximally entangled state of the bipartite square bit exhibits Hardy's nonlocality argument with

[1]A state Φ^{AB} is called an inner product state if Φ^{AB} is symmetric, and positive semi-definite, *i.e.* $(e \otimes e)(\Phi^{AB}) \geq 0,\ \forall\ e \in V^{\star}$ [34].

success probability $1/2$, which is, in fact, the maximum Hardy's success among any no-signaling correlations. In the following, we prove a generic result that the maximally entangled state of any bipartite even gon depicts Hardy's type of nonlocality, albeit with decreasing success probability.

At this point, it is worth mentioning that the maximally entangled state of the quantum two-qubit system fails to exhibit Hardy's nonlocality behaviour [87]. In this sense, odd gons are closer to quantum than even gons as the maximally entangled states of the former do not depict Hardy nonlocality while the latter do.

Theorem 9. *The maximally entangled state Φ_J of bipartite even-gons (with $n \geq 4$) exhibits Hardy's non-locality argument with the success probability given by* $\sin^2 \frac{\pi}{n}$.

The proof of the above two theorems is included in the appendix [B.2]. The variation of Hardy's success probability for different even gons is shown in Fig. [4.3].

4.3.2 Hardy's nonlocality for non-maximally entangled states

For the bipartite pentagon case, we only have two inequivalent classes of pure entangled states, the state Φ_J of Eq.(4.2) and the state Φ_H of Eq.(4.5). As already established in Theorem [8], the state Φ_J cannot result in any correlation exhibiting Hardy's nonlocality. So the natural question arises whether the state Φ_H can lead to such a correlation. Interestingly we find that that state Φ_H indeed exhibits Hardy's nonlocality. If we consider two incompatible measurements $\mathbf{M}_1 \equiv \{e_1, \bar{e}_1\}$ and $\mathbf{M}_2 \equiv \{e_5, \bar{e}_5\}$ on Alice's part and two incompatible measurements $\mathbf{N}_1 \equiv \{e_1, \bar{e}_1\}$ and $\mathbf{N}_2 \equiv \{\bar{e}_2, e_2\}$ on Bob's part, the resulting correlation depicts Hardy's nonlocality. Denoting the outcome corresponding to the first effect as $+1$ and the outcome corresponding to the second one as -1, the correlation obtained from these choices of measurements reads as

$$\mathbf{P} \equiv \begin{array}{c||c|c|c|c|} & (+1,+1) & (+1,-1) & (-1,+1) & (-1,-1) \\ \hline \mathbf{M}_1\mathbf{N}_1 & 1-\frac{4\sqrt{5}}{10} & \frac{7\sqrt{5}}{10}-\frac{3}{2} & \frac{7\sqrt{5}}{10}-\frac{3}{2} & 3-\sqrt{5} \\ \hline \mathbf{M}_1\mathbf{N}_2 & 0 & \frac{3\sqrt{5}}{10}-\frac{1}{2} & \frac{\sqrt{5}}{10}+\frac{1}{2} & 1-\frac{4\sqrt{5}}{10} \\ \hline \mathbf{M}_2\mathbf{N}_1 & 0 & \frac{2\sqrt{5}}{10} & \frac{3\sqrt{5}}{10}-\frac{1}{2} & \frac{3}{2}-\frac{5\sqrt{5}}{10} \\ \hline \mathbf{M}_2\mathbf{N}_2 & \frac{3\sqrt{5}}{10}-\frac{1}{2} & \frac{1}{2}-\frac{\sqrt{5}}{10} & 1-\frac{2\sqrt{5}}{10} & 0 \\ \hline \end{array} . \tag{4.7}$$

Important to note that the success probability of Hardy's argument in this case is $P(+,+|\mathbf{M}_1\mathbf{N}_1) = 1-\frac{4\sqrt{5}}{10} \approx 0.1056$, which is strictly larger than the corresponding optimal quantum value $\frac{5\sqrt{5}-11}{2} \approx 0.0902$. Therefore, this particular correlation is beyond quantum in nature, although its CHSH value is strictly less than the Cirel'son's value. The success probability 0.1056 turns out to be optimal in pentagon theory. For the hexagon case, we find that all the six different classes of states depicts Hardy's nonlocality. The choices of measurements and the corresponding Hardy's success probabilities are listed in Table [4.1].

4.3.3 Mixed states exhibiting Hardy Nonlocality

We will now consider the possibilities of the Hardy-type nonlocality for those preparation devices that produce a pure entangled g-bit [2] or a pure product g-bit with a pre-declared ignorance, and hence result in a mixed preparation. In quantum theory, such a preparation devise which prepares a two-qubit pure entangled state and a two-qubit pure product state with some predefined ignorance doesn't exhibit Hardy Nonlocality. (Technically any two-qubit mixed state does not exhibit Hardy-type nonlocality [88]. But here we are interested in these special preparation devices as they can help us study some topological properties of the underlying GPT). However, we will show this is not the case in polygonal GPT models. In the following, we will identify the *preparation-measurement reciprocity* [95] as the salient feature of quantum theory, which causes such a difference. From now on, by the phrase "mixed entangled state"

[2]Analogous to the terms c-bit and qubit for two-level classical and quantum systems respectively, here we adopt the phrase *g-bit* to denote the elementary system of a GPT, with (Operational Dimension) OD=2.

State	Alice's Measurements	Bob's Measurements	Hardy's Success
Φ_I	$\mathbf{M}_1 = \{e_1, e_4\}$ $\mathbf{M}_2 = \{e_2, e_5\}$	$\mathbf{N}_1 = \{e_6, e_3\}$ $\mathbf{N}_2 = \{e_5, e_2\}$	1/4
Φ_{II}	$\mathbf{M}_1 = \{e_4, e_1\}$ $\mathbf{M}_2 = \{e_2, e_5\}$	$\mathbf{N}_1 = \{e_4, e_1\}$ $\mathbf{N}_2 = \{e_2, e_5\}$	1/20
Φ_{III}	$\mathbf{M}_1 = \{e_3, e_6\}$ $\mathbf{M}_2 = \{e_2, e_5\}$	$\mathbf{N}_1 = \{e_1, e_4\}$ $\mathbf{N}_2 = \{e_6, e_3\}$	1/16
	$\mathbf{M}_1 = \{e_4, e_1\}$ $\mathbf{M}_2 = \{e_2, e_5\}$	$\mathbf{N}_1 = \{e_1, e_4\}$ $\mathbf{N}_2 = \{e_5, e_2\}$	1/8
Φ_{IV}	$\mathbf{M}_1 = \{e_3, e_6\}$ $\mathbf{M}_2 = \{e_2, e_5\}$	$\mathbf{N}_1 = \{e_4, e_1\}$ $\mathbf{N}_2 = \{e_3, e_6\}$	1/28
	$\mathbf{M}_1 = \{e_4, e_1\}$ $\mathbf{M}_2 = \{e_2, e_5\}$	$\mathbf{N}_1 = \{e_4, e_1\}$ $\mathbf{N}_2 = \{e_2, e_5\}$	1/14
Φ_V	$\mathbf{M}_1 = \{e_4, e_1\}$ $\mathbf{M}_2 = \{e_3, e_5\}$	$\mathbf{N}_1 = \{e_3, e_6\}$ $\mathbf{N}_2 = \{e_2, e_5\}$	1/28
	$\mathbf{M}_1 = \{e_4, e_1\}$ $\mathbf{M}_2 = \{e_2, e_5\}$	$\mathbf{N}_1 = \{e_4, e_1\}$ $\mathbf{N}_2 = \{e_2, e_5\}$	1/14
Φ_{VI}	$\mathbf{M}_1 = \{e_2, e_5\}$ $\mathbf{M}_2 = \{e_1, e_4\}$	$\mathbf{N}_1 = \{e_2, e_5\}$ $\mathbf{N}_2 = \{e_1, e_4\}$	1/8

TABLE 4.1 Measurement choices for Alice and Bob and the corresponding Hardy's success probabilities for the six different classes of entangled states in bipartite hexagon theory.

we would like to mean a convex combination of pure entangled state along with a product g-bit. For the sake of completeness, let us begin with a definition of preparation-measurement reciprocity in quantum theory.

Definition 32 (Preparation-measurement reciprocity [95]). *For every pure quantum preparation $\omega_\psi = |\psi\rangle\langle\psi|$ the corresponding effect, i.e., $e_\psi = |\psi\rangle\langle\psi|$ clicks certainly whenever the measurement $\mathbf{M} := \{e_\psi, u - e_\psi\}$ is performed. Furthermore, ω_ψ is the only preparation which passes the effect e_ψ with certainty.*

The notion can readily be extended in GPTs. We say a GPT satisfies 'preparation-measurement reciprocity' if for every pure state ω there exists a unique extremal effect e_ω that filters the state ω with certainty. Also note that all such GPT models can be assigned with a finite dimensional linear vector space and hence the effects, i.e., the linear functional on this vector space allows a one-to-one correspondence with the state vectors, which is referred as *weak self-duality* [34, 96]. This, in turn, further strengthens the condition of preparation-measurement reciprocity in such GPT models.

Definition 33 (Hardy-type local correlation). *A correlation is called Hardy-type local if it is local and it satisfies all the zero constraints (last three equalities) in Eq.(2.53).*

Definition 34 (Trivial Hardy-local theory). *A bipartite theory is said to be trivial Hardy-local if no product state results in a Hardy-type local correlation under incompatible measurements performed on the subsystems.*

With these definitions, we are now in a position to prove an important result. The proofs are detailed in [B.3].

Lemma 2. *The bipartite composition of a GPT will be a 'trivial Hardy-local theory' whenever its local parts have $OD = 2$ and satisfy preparation-measurement reciprocity.*

With the help of the above Lemma, we will finally conclude that,

Theorem 10. *A mixed preparation device, for a theory (with OD=2 for each subsystem) obeying preparation-measurement reciprocity on each part, does not exhibit Hardy nonlocality.*

Now, we will show that bipartite compositions of all the discrete operational models with an even number of pure states and specifically the pentagon model exhibit Hardy-type nonlocality when a suitably chosen pure product preparation probabilistically sampled with a pure entangled preparation. Since, for all such theories the OD is exactly 2, we can identify the absence of preparation-measurement reciprocity in their topology. We conjecture the same feature also holds for any odd n-gon models, however, due to the numerical limitations we are bound with $n = 5$ case only.

Theorem 11. *For every even (n)-gon theory and $\forall \varepsilon \in (0,1]$, there exists a class of mixed entangled states $W_\varepsilon = \varepsilon\Phi_J + (1-\varepsilon)\omega_i \otimes \omega_j$ exhibiting Hardy nonlocality.*

In a similar spirit, it is possible to show that the mixed state of odd-gon theories can also exhibit Hardy's nonlocality. In the following, we give a proof for the bipartite pentagon theory.

Theorem 12. *For every value of $\varepsilon \in (0,1]$, the mixed entangled state $W_\varepsilon = \varepsilon\Phi_H + (1-\varepsilon)\omega_i \otimes \omega_j$ exhibits Hardy-type nonlocality for suitable choice of measurement, whenever $\omega_i \otimes \omega_j \in \{\omega_3 \otimes \omega_4, \omega_3 \otimes \omega_5, \omega_4 \otimes \omega_3, \omega_4 \otimes \omega_4, \omega_5 \otimes \omega_3\}$.*

While from the perspective of Bell-nonlocality, the polygonal state spaces exhibit no characteristic distinction (except the quantitative bounds) from their continuous counterpart (Quantum theory), Theorem [11] and [12] exhibit such a distinction for Hardy-type nonlocal arguments. However, the signature of such a difference vanishes considering the bipartite compositions of higher quantum systems. In particular, for higher dimensional Quantum theory, there are incompatible local measurements with the proper choice of separable bipartite state, which can generate any of the extreme local correlations, and hence there are mixed entangled states depicting Hardy-type nonlocal arguments. This, in turn, directs towards the state space topology of the qubit system and the continuity therein to demonstrate it as unique among the possible two-dimensional state space structures.

4.4 Inequivalence of entanglement and B-CHSH nonlocality in polygon models

While all bipartite quantum pure states exhibit nonlocality [97], the pure states are too idealistic when experimental situations are considered. So naturally the question arises whether mixed states exhibit such nonlocal behaviour. A particular family that is of interest to us is the Werner class of states

$$W_p = p\,|\psi^-\rangle_{AB}\langle\psi^-| + (1-p)\frac{\mathbb{I}}{2}\otimes\frac{\mathbb{I}}{2}, \tag{4.8}$$

where $|\psi^-\rangle := (|01\rangle - |10\rangle)/\sqrt{2} \in \mathbb{C}^2\otimes\mathbb{C}^2$ and $p\in[-1/3,1]$. In particular, for $p\in[0,1]$ the state can be thought of as a statistical mixture of the singlet state and white noise. Straightforward calculation yields the state W_p is entangled for $p>\frac{1}{3}$ and violates CHSH inequality for $p>\frac{1}{\sqrt{2}}$. In a seminal result, Werner established that for $\frac{1}{3}<p\leq\frac{1}{2}$ the statistics obtained from the state W_p through local projective measurements can be explained by local hidden variable model [98]. Later Barrett extended this model for arbitrary local measurement for the parameter range $p\leq\frac{5}{12}$ [99] (see also [100]). This result is quite important as it establishes that entanglement and nonlocality as two inequivalent notions. A similar question one can ask in polygon theories. Our next result partially addresses this question.

Theorem 13. *For all the theories where $n>4$, there exists a class of mixed entangled states that does not violate CHSH inequality.*

Proof. Consider the class of states, $W_p^O := p\Phi_J + (1-p)u\otimes u$, where $p \in [0,1]$ and Φ_J is the state given in Eq. (4.2). Clearly, W_p^O is a mixed state whenever $p \in [0,1)$.

For the odd-gons, the expectation value of any measurement $\langle \mathscr{M}_i \mathscr{N}_j \rangle$ on the maximally mixed state $u\otimes u$ reads as $\left(\frac{r_n^2-1}{r_n^2+1}\right)^2$. For the state W_p^O the maximum value of Bell-CHSH expression becomes $\mathbb{B}_{\max}(W_p^O) = p\mathbb{B}_{\max}^{(n)} + 2(1-p)\left(\frac{r_n^2-1}{r_n^2+1}\right)^2$, where $\mathbb{B}_{\max}^{(n)}$ is the maximum Bell-CHSH value obtained from the n-gonal maximum entangled state Φ_J. Denoting the range of the parameter p_{NL} of the state W_p^O showing Bell-CHSH nonlocality we have

$$p_{NL} > 8r_n^2 \left[\mathbb{B}_{\max}^{(n)} \left(r_n^2+1\right)^2 - 2\left(r_n^2-1\right)^2\right]^{-1}. \tag{4.9}$$

We now proceed to find the range of the parameter p of the state W_p^O for which the state is entangled. Note that, unlike quantum theory, in this model, we do not have any criterion like negative partial transposition (NPT) [101, 102] that can detect the entanglement of a state. However, if we can find an effect that is entangled and yields a negative probability on some state, then by definition, the state must be entangled[3]. This is because any product state on an entangled effect always gives a non-negative probability. For the odd-gon theory, it has been shown that the effects E_{ab} and $\bar{E}_{ab} := u\otimes u - E_{ab}$ are entangled [106], where

$$E_{ab} = \frac{1}{1+r_n^2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

A straightforward calculation yields $\mathrm{Tr}[\bar{E}_{ab}^T W_p^O] = p\left(\frac{r_n^2-2}{1+r_n^2}\right) + (1-p)\frac{r_n^2}{1+r_n^2}$. Denoting the range of parameter of the state W_p^O as p_E for which the state must be entangled we have

$$\mathrm{Tr}[\bar{E}_{ab}^T W_{p_E}^O] < 0 \implies p_E > \frac{r_n^2}{2}. \tag{4.10}$$

[3]At this point, an observant reader should note that in quantum theory all the entangled states yield non-negative probability on all the entangled effects. This is due to the fact that state and effect cones are self-dual. However, in abstract GPT, this might not be the case, which in turn results in different consistent compositions for the same elementary systems. At this point the Refs.[46, 36, 14, 103–105] are worth mentioning.

Comparing Eq. (4.9) and Eq. (4.10), it is evident that $p_E < p_{NL}\ \forall$ odd n. Therefore within the range $p_E \leq p < p_{NL}$ the state $W^O_{p_E}$ is entangled but it does not violate Bell-CHSH inequality.

For the even gon theory we consider the state $W^E_p := p\Phi_J + (1-p)u\otimes u$, where $p \in [0,1]$ and Φ_J is the state given in Eq. (4.3). Noting the fact that $\langle \mathscr{M}_i \mathscr{N}_j\rangle$ on $u\otimes u$ turns out to be zero in this case and using the entangled effect

$$E_{ab} = \frac{1}{2}\begin{pmatrix} -\cos\frac{\pi}{n} & -\sin\frac{\pi}{n} & 0 \\ \sin\frac{\pi}{n} & -\cos\frac{\pi}{n} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

identified in [?], a similar calculation yields

$$p_{NL} > \frac{2}{\mathbb{B}^{(n)}_{\max}}, \ \ \& \ \ p_E > \frac{1}{2}, \tag{4.11}$$

where $\mathbb{B}^{(n)}_{\max}$ is the maximal Bell-CHSH value for the state Φ_J. Since for all the even gons $\mathbb{B}^{(n)}_{\max} < 4$, whenever $n > 4$ [34], therefore within the range $p_E \leq p < p_{NL}$ the state W^E_p does not violate Bell-CHSH inequality although it is entangled. This completes the proof. □

Important to note that, for $n = 4$, we Have $\mathbb{B}^{(4)}_{\max} = 4$, which thus leads to the fact that all the entangled states in this theory are Bell-CHSH nonlocal.

4.5 Discussions

Our study focused on the classification of extreme bipartite states within locally regular polygonal systems, with a particular emphasis on their nonlocal characteristics. Utilizing the methodology described in Appendix [B.1], we established that the bipartite state space of the square system admits a unique class of entangled states. In contrast, for local pentagonal and hexagonal systems, we identified two and six distinct categories of entangled states, respectively. Furthermore, we observed that while the states given by Eq. (4.2) and Eq. (4.3), along with their equivalent representations, exhibit characteristics akin to maximally entangled two-qubit states, the newly identified classes more closely resemble non-maximally entangled two-qubit states.

To further investigate the nonlocal properties of polygonal theories, we employed Hardy's nonlocality argument. Notably, unlike the case of bipartite quantum theory with local operational dimension $\mathrm{OD} = 2$, we determined that in even-gon theories, the maximally entangled states satisfy Hardy's argument with a success probability of $\sin^2(\frac{\pi}{n})$. Moreover, as the number of vertices in even-gon systems increases, the corresponding success probability diminishes. Conversely, we did not observe an analogous nonlocal effect in bipartite odd-gon systems with maximally entangled states. However, within the bipartite pentagonal theory, we identified an enhanced degree of nonlocality in the non-maximally entangled states Φ_H, exceeding the maximum success probability predicted by quantum theory.

Beyond the hexagonal state space, our findings raise several open questions for future investigation. While we propose a systematic approach for identifying distinct classes of extreme entangled states, the computational complexity of this method renders it impractical for arbitrarily large higher-gon theories. Consequently, the development of a more efficient methodology applicable to these scenarios is of paramount importance. Additionally, the characterization of Hardy nonlocality in higher odd-gon theories remains an open problem.

Chapter 5

Harnessing Indefinite Composition of Spacetime Regions to Access Locally Inaccessible Data

5.1 Introduction

In the preceding two chapters, we examined the effects of composition on both timelike and spacelike correlations. However, seminal results such as those presented in [11] reveal that more general spacetime correlations can emerge when considering the broader framework of process matrices [2.4]. Specifically, there exist correlations for which any causal interpretation becomes impossible. These correlations arise from quantum processes which are conventionally termed as *indefinitely ordered processes*.

In this work, we present an information-theoretic perspective on such nontrivial processes. We define a multiplayer task wherein each player seeks to retrieve a hidden piece of data from a multipartite quantum state. The data is encoded in such a manner that no player, through local operations, can access any information regarding the hidden data. The objective of the players is to collaborate in order to retrieve their respective pieces of information.

We demonstrate that players embedded within an indefinitely ordered spacetime generally achieve better performance compared to those situated within a definite spacetime. This task, which we refer to as the *Data Retrieval Task*, is described more formally in the following section.

5.2 Data Retrieval task (DR task)

In this section, we formally define the Data Retrieval (DR) task (see Fig.C.1). A referee distributes an N-dit string message $\mathbf{x} = x_1x_2\cdots x_N \in \{0,1,\ldots,d-1\}^N$

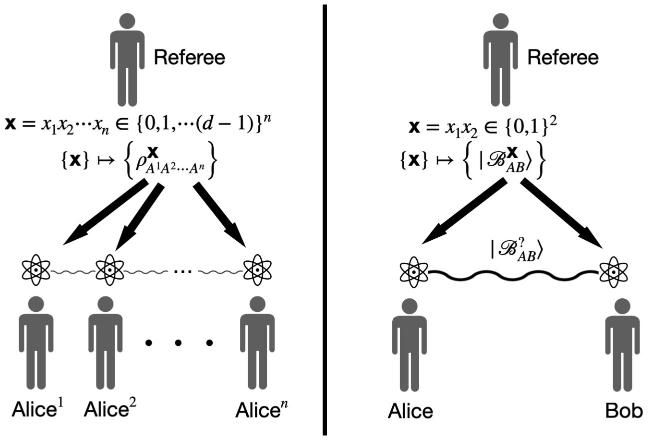

FIG. 5.1 Data Retrieval (DR) task involving n parties (left). Referee encodes the stings $\mathbf{x} \equiv x_1x_2\cdots x_n \in \{0,\cdots,d-1\}^{\times n}$ into n-partite quantum states $\rho^{\mathbf{X}}_{A^1A^2\cdots A^n} \in \mathscr{D}(H_{A^1} \otimes \cdots \otimes H_{A^n})$ and distributes the subsystems to the respective parties. Local marginals being independent of $\mathbf{x}$ ensure that none of the parties can reveal any information about $\mathbf{x}$ on their own. However, collaboration among themselves might be helpful to retrieve their respective messages. (Right) Data Retrieval task with two-qubit Bell states encoding.

among N parties $\{\text{Alice}^k\}_{k=1}^N$, each residing in a spatially separated laboratory. The referee aims to ensure that no individual party can obtain any information about the string $\mathbf{x}$ on their own. We refer to this requirement as the *hiding condition*. To meet this condition, the referee encodes the message into an ensemble of N-partite quantum states $\{\rho^{\mathbf{x}}_{A^1\cdots A^N}\}_{\mathbf{x}} \subset \mathscr{D}(\otimes_{k=1}^N H_{A^k})$, and sends the k-th part of the encoded state to Alice^k. The hiding condition requires that the individual marginals of the encoded state be independent of $\mathbf{x}$, *i.e.*,

$$\rho^{\mathbf{x}}_{A^k} := \mathrm{Tr}_{A^1\cdots A^N\setminus A^k}\left[\rho^{\mathbf{x}}_{A^1\cdots A^N}\right] = \rho_{A^k},\ \forall\ \mathbf{x}, k, \tag{5.1}$$

where $\mathrm{Tr}_{A^1\cdots A^N\setminus A^k}(\cdot)$ denotes partial trace over all the subsystems except the k^{th} one. Note that ρ_{A^k} can, in general, differ from $\rho_{A^{k'}}$ for $k \neq k'$. The goal of each player is to retrieve their corresponding dit value, i.e., Alice^k aims to retrieve the value x_k.

To this end, the parties can adopt different collaboration strategies depending on the resources that are available to them. If the parties are allowed to communicate only classical information among each other, then the parties, performing local quantum operations on their respective shares of the composite system, can resort to the operational paradigm of local operation and classical communication (LOCC), which naturally arises in the resource theory of quantum entanglement [84]. On the other hand, replacing classical communication lines by quantum channels one obtains a stronger form of collaboration: local operation and quantum communication (LOQC). It is important to note that if the encoded states are not mutually orthogonal, then the parties cannot perfectly determine their respective messages even with LOQC collaboration. In other words, the perfect success requires the encoded states to be mutually orthogonal, *i.e.*, $\mathrm{Tr}[\rho^{\mathbf{x}}\rho^{\mathbf{x}'}] = 0 \ \forall \ \mathbf{x} \neq \mathbf{x}'$.

Notably, both in LOQC and in LOCC collaborations, the protocol goes in multi rounds [107]. At this stage, one may impose restriction on the rounds of communication. For instance, consider the single-opening setup as defined in Section[2.4.4]: at a given run of the task, each party's laboratory opens only once, during which they can receive a system in their laboratory, implement an operation on it, and send it out of the laboratory, with each step occurring only once. Such a collaboration scenario is considered while developing the process matrix framework [11]. In this single-opening setup, we particularly focus whether causally inseparable processes could be advantageous over causally separable processes in DR tasks. To address this question, we consider an explicit example of such a task for the simplest case.

5.2.1 Data Retrieval from Bell States (DR-B task)

Consider the simplest case of DR task with $d = 2$ and $N = 2$. Referee encodes the strings $\mathbf{x} = x_1x_2 \in \{0,1\}^2$ into maximally entangled basis of $\mathbb{C}^2 \otimes \mathbb{C}^2$ system:

$$\mathbf{x} \mapsto |\mathscr{B}^{\mathbf{x}}\rangle_{AB} := \frac{1}{\sqrt{2}}(|0x_1\rangle + (-1)^{x_2}|1\bar{x}_1\rangle)_{AB}, \tag{5.2}$$

where $\{|0\rangle, |1\rangle\}$ represents the computational basis. Accordingly, the encoded states are distributed between Alice and Bob. Clearly the hiding condition is satisfied,

$$\rho^{\mathbf{x}}_{A(B)} = \mathrm{Tr}_{B(A)}[|\mathscr{B}^{\mathbf{x}}\rangle_{AB}\langle\mathscr{B}^{\mathbf{x}}|] = \mathbf{I}_{A(B)}/2, \ \forall \ \mathbf{x}. \tag{5.3}$$

Since Bell states are used for encoding, we call this task Locally Inaccessible Data Retrieval task from Bell states (DR-B). Alice and Bob have to guess the bit value x_1 and x_2, respectively. Denoting their respective guesses 'a' and 'b', the success of the task reads as

$$P^{\text{DR-B}}_{succ} = \sum_{x_1,x_2=0}^{1} \frac{1}{4} P(a = x_1, b = x_2 | \mathscr{B}^{\mathbf{x}}_{AB}). \tag{5.4}$$

x_1x_2	$\lvert\mathscr{B}^{\mathbf{x}}\rangle$	Alice's outcome	Bob's outcome	a	b	Status
00	$\lvert\phi^+\rangle$	up	down	0	0	success
		down	up	1	1	failure
01	$\lvert\phi^-\rangle$	up	up	0	1	success
		down	down	1	0	failure
10	$\lvert\psi^+\rangle$	up	up	0	1	failure
		down	down	1	0	success
11	$\lvert\psi^-\rangle$	up	down	0	0	failure
		down	up	1	1	success

TABLE 5.1 Protocol for DR-B task as discussed in Proposition 3. Success probability turns out to be $P^{\text{DR-B}}_{succ} = 1/2$.

Since the local parts of the encoded states do not contain any information of $\mathbf{x}$, without any collaboration a random guess by Alice of Bob will yield $P^{\text{DR-B}}_{succ} = 1/4$. However, they can come up with a better strategy even without any collaboration.

Proposition 3. *Without any collaboration Alice and Bob can achieve the success* $P^{DR\text{-}B}_{succ} = 1/2$.

Proof. Their protocol goes as follows: both the players performs σ^2 (*i.e.* Pauli-Y) measurement on their part of the encoded state received from the referee. Alice answers $a = 0$ ($a = 1$) for 'up' ('down') outcome, while Bob answers $b = 1$ ($b = 0$) for 'up' ('down') outcome. The claimed success probability follows from Table 5.1. □

Proposition 4. *Under LOCC collaboration* $P^{DR\text{-}B}_{succ} \leq 1/2$.

Proof. The proof simply follows from the optimal probability of distinguishing Bell states under LOCC [108–110]. □

Proposition 5. *Under LOQC collaboration* $P^{DR\text{-}B}_{succ} = 1$.

Proof. Alice sends her part of the encoded state to Bob through a perfect qubit channel; Bob performs a Bell basis measurement to retrieve both x_1 & x_2, and classically communicates back x_1 to Alice. □

Notably, Proposition 5 holds true for any DR task whenever the encoded states are mutually orthogonal. We will now consider the single opening scenario. Within this setup, we start by establishing a bound on DR-B success whenever the players are embedded in a definite causal structure.

Proposition 6. *In the single-opening setup* $P^{DR\text{-}B}_{succ} \leq 1/2$*, whenever the players are embedded in a definite causal structure.*

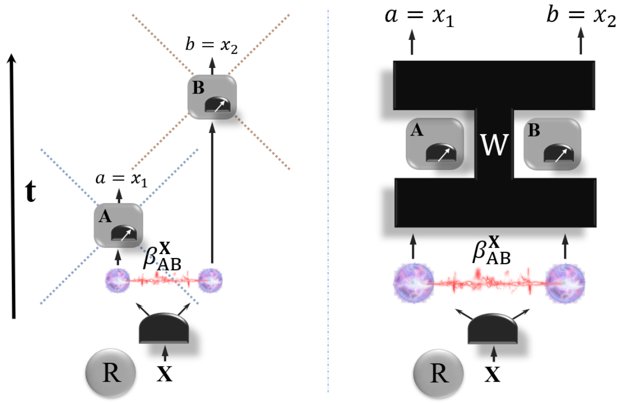

FIG. 5.2 DR-B task: Referee encodes the string **x** in four Bell states. The players' strategies to guess their respective bits in single-opening setup are shown above. Left one depicts the scenario when they are embedded in definite causal structure (here Alice is in the causal past of Bob). Right one depicts the scenario when they share some indefinite causal process W.

Proof. (Heuristic argument) Assume that Alice is in the causal past of Bob. Thus communication from Alice to Bob is possible, but not in other direction, *i.e.*, they can share a process of type $W^{A\prec B}_{A_IA_OB_IB_O}$, along with the given encoded state $\mathscr{B}^{\mathbf{x}}_{AB} := |\mathscr{B}^{\mathbf{x}}\rangle_{AB}\langle\mathscr{B}^{\mathbf{x}}|$. Marginal of the encoded state being independent of **x**, Alice cannot obtain any information about **x** from $W^{A\prec B}_{A_IA_OB_IB_O} \otimes \mathscr{B}^{\mathbf{x}}_{AB}$. Therefore she can at best randomly guess the bit value of x_1, while Bob can identify both x_1 and x_2 perfectly. Therefore success probability in this case is upper bounded by $1/2$. Similar argument holds for the processes of types $W^{B\prec A}_{A_IA_OB_IB_O}$, and also for $W^{B\nprec\nsucc A}_{A_IA_OB_IB_O}$ (see Proposition [3]). Finally note that any causally separable process can be expressed as Eq.(C.25), and hence the claim follows from convexity. □

As we will see a more formal proof of Proposition [6] can be obtained as a consequence of one of our core results established in the next section (see Corollary 2).

5.3 Advantage of causal inseparability in DR-B task

Here we will show that Alice and Bob can obtain advantage in DR-B task when they share causally inseparable processes (see Fig.5.2). To this aim we proceed to establish a generic connection between the success probabilities of two independent tasks - the GYNI game and the DR-B task.

Theorem 14. *A success probability $P^{DR\text{-}B}_{succ} = \mu$ in DR-B task is achievable if and only if the same success is achievable in GYNI game, i.e., $P^{GYNI}_{succ} = \mu$.*

The proof is divided into two parts:

(i) *only if* part: $P_{succ}^{\text{DR-B}} = \mu$ ensures a protocol for GYNI game yielding success probability $P_{succ}^{\text{GYNI}} = \mu$. The formal proof provided in Appendix [C.1] proceeds by observing that the GYNI game can be mapped to the DR-B task if Alice and Bob share an additional Bell state, denoted by $\mathscr{B}_{AB}^{00}$. Let i_1 and i_2 be the respective coin toss outcomes for Alice and Bob in the GYNI game, and they aim to guess i_2 and i_1, respectively. By applying appropriate local unitary operations—specifically, the Pauli gates $\mathbb{Z}_A^{i_1}$ and $\mathbb{X}_B^{i_2}$—to the shared Bell state $\mathscr{B}_{AB}^{00}$, the initial state is transformed into the Bell state $\mathscr{B}_{AB}^{i_2 i_1}$. Following this step Alice and Bob simply employ the protocol that guarantees a success probability $P_{succ}^{\text{DR-B}} = \mu$ in the DR-B task.

(ii) *if* part: $P_{succ}^{\text{GYNI}} = \mu$ ensures a protocol for DR-B task yielding success probability $P_{succ}^{\text{DR-B}} = \mu$. Here we provide the explicit details of the protocol while formal proof is provided in Appendix [C.1]. It is well known that the set of Bell states $\mathscr{B}_{AB}^{x_1 x_2}$ cannot be perfectly distinguished using only local operations and classical communication (LOCC); in other words, the variables x_1 and x_2 cannot be simultaneously determined by local means. However, perfect discrimination becomes possible if Alice and Bob are given access to an additional maximally entangled Bell state, denoted by $\mathscr{B}_{A'B'}^{00}$. We exploit this fact by first transferring the information of one of the variables (say, x_1) from the original Bell state into the ancillary Bell state followed by interchanging x_1 and x_2 in the original state, i.e.,

$$\mathscr{B}_{A'B'}^{00} \otimes \mathscr{B}_{AB}^{x_1 x_2} \rightarrow \mathscr{B}_{A'B'}^{x_1 0} \otimes \mathscr{B}_{AB}^{x_2 x_1}.$$

This can be achieved by applying a suitable local unitary transformation. Alice and Bob then perform measurements in the computational basis on both Bell pairs. Let the measurement outcomes be denoted by $u, u' \in \{0,1\}$ for Alice and $v, v' \in \{0,1\}$ for Bob. Due to the correlation properties of the Bell states, these outcomes satisfy:

$$u \oplus v = x_2 \quad \text{and} \quad u' \oplus v' = x_1.$$

Therefore, if Alice and Bob wish to learn the values of x_1 and x_2, respectively, it suffices for Bob to send v' to Alice, and for Alice to send u to Bob. This task is precisely is the GYNI game. Consequently, if a success probability

μ is achievable in the GYNI game, then the same success probability μ can be achieved in the corresponding DR-B task via this construction.

To demonstrate that indefiniteness of processes is necessary to obtain the aforementioned advantage, we show below that any causally separable process (see Section.[27]) cannot yield a non-trivial advantage in the DR-B task. In fact, we establish that even extensibly causal processes (Section.[2.4.5]) which are a strict super set of causally separable processes are insufficient to provide any advantage in the DR-B task.

Corollary 2. *Any extensibly causal process cannot provide non trivial advantage in the DR-B task.*

Proof. We start by assuming that an extensibly causal process W^{EC} can provide nontrival success $\mu > 1/2$ in the DR-B task. Now from Theorem [14] its clear that $W^{EC} \otimes \phi^+$ can win the GYNI game with success probability $\mu > 1/2$. But this is in contradiction to the definition of extensibly causal processes. This completes the proof. Note that this also serves as a formal proof for Proposition 6. □

Theorem [14] can be easily generalized to bipartite DR tasks with $d > 2$ (*i.e.*, encoding the strings $\mathbf{x} = x_1x_2 \in \{0, 1, \cdots, d-1\}^2$ into a maximally-entangled basis of $\mathbb{C}^d \otimes \mathbb{C}^d$) and to higher-input GYNI games. The proof is provided in Appendix [C.2]. In the following section, we proceed to analyze the necessary condition for process matrices to be useful in the DR-B task.

5.4 Necessary condition for Quantum Processes to be useful in DR-B

Theorem [14] establishes that any bipartite process yielding non-trivial success in the GYNI game, when combined with an additional maximally entangled state, achieves similar success in the DR-B task. It is now natural to ask whether all such causally inseparable processes are advantageous in the DR-B task on their own. Interestingly, we answer this question in negative. In the following we show that a given process matrix needs to be of some particular form to be useful in DR-B task.

Theorem 15. *A bipartite process matrix, $W_{A_IA_OB_IB_O}$ yielding a nontrivial success in the DR-B task (i.e., $P^{DR\text{-}B}_{succ} > 1/2$) must have negative partial transposition across $(A_IA_O)|(B_IB_O)$ bipartition.*

Proof. Consider Alice and Bob share a bipartite process $W_{A_IA_OB_IB_O}$ for the DR-B task. Given the encoded state $\mathscr{B}^{\mathbf{x}}_{AB} := |\mathscr{B}^{\mathbf{x}}\rangle_{AB}\langle\mathscr{B}^{\mathbf{x}}|$, Alice and Bob respectively implement two-outcome instruments $\{N^a_{AA_I\to A_O}\}^1_{a=0}$ and $\{M^b_{BB_I\to B_O}\}^1_{b=0}$ on their respective shares of the joint process $\mathscr{B}^{\mathbf{x}}_{AB}\otimes W_{A_IA_OB_IB_O}$, and give the classical outcomes a and b as their respective guesses for x_1 and x_2. The success probability is then given by

$$P^{\text{DR-B}}_{succ} = \frac{1}{4}\sum_{x_1,x_2=0}^{1} p(a=x_1, b=x_2|\mathscr{B}^{x_1x_2}_{AB}) \tag{5.5a}$$

$$= \text{Tr}\Big[(\mathscr{B}^{\mathbf{x}}_{AB}\otimes W_{A_IA_OB_IB_O})\Big\{\mathbf{id}_{AA_I}\otimes N^{a=x_1}_{A'A'_I\to A_O}\otimes \mathbf{id}_{BB_I}\otimes M^{b=x_2}_{B'B'_I\to B_O}(\tilde{\phi}^+_{AA_IA'A'_I}\otimes\tilde{\phi}^+_{BB_IB'B'_I})\Big\}\Big]. \tag{5.5b}$$

Here, $p(a,b|\mathscr{B}^{\mathbf{x}}_{AB}) := p(a=x_1,b=x_2|\mathscr{B}^{x_1x_2}_{AB})$ is the probability of getting outcomes $a=x_1$ and $b=x_2$, with $\tilde{\phi}^+ := |\tilde{\phi}^+\rangle\langle\tilde{\phi}^+|$. Accordingly we have

$$p(a,b|\mathscr{B}^{\mathbf{x}}_{AB}) = \text{Tr}\Big[\Big\{\mathbf{id}_{ABA_IB_I}\otimes N^{(*)a}_{A_O\to A'A'_I}\otimes M^{(*)b}_{B_O\to B'B'_I}(\mathscr{B}^{\mathbf{x}}_{AB}\otimes W_{A_IA_OB_IB_O})\Big\}(\tilde{\phi}^+_{AA_IA'A'_I}\otimes\tilde{\phi}^+_{BB_IB'B'_I})\Big], \tag{5.6}$$

where $N^{(*)}$ and $M^{(*)}$ are the dual maps[1] of N and M, respectively. Defining, $\tilde{W}^{a,b}_{A_IA'A'_IB_IB'B'_I} := \mathbf{id}_{A_IB_I}\otimes N^{(*)a}_{A_O\to A'A'_I}\otimes M^{(*)b}_{B_O\to B'B'_I}(W_{A_IA_OB_IB_O})$, we obtain

$$p(a,b|\mathscr{B}^{\mathbf{x}}_{AB}) = \text{Tr}\Big[(\mathscr{B}^{\mathbf{x}}_{AB}\otimes\tilde{W}^{a,b})(\tilde{\phi}^+_{AA'}\otimes\tilde{\phi}^+_{A_IA'_I}\otimes\tilde{\phi}^+_{BB'}\otimes\tilde{\phi}^+_{B_IB'_I})\Big]$$
$$= \text{Tr}_{A,A',B,B'}\Big[(\mathscr{B}^{\mathbf{x}}_{AB}\otimes\chi^{a,b}_{A'B'})(\tilde{\phi}^+_{AA'}\otimes\tilde{\phi}^+_{BB'})\Big]$$
$$= \text{Tr}_{A,B}\Big[\mathscr{B}^{\mathbf{x}}_{AB}\Pi^{a,b}_{AB}\Big], \tag{5.7a}$$

$$\Pi^{a,b}_{AB} := \text{Tr}_{A',B'}[(\mathbb{I}_{AB}\otimes\chi^{a,b}_{A'B'})(\tilde{\phi}^+_{AA'}\otimes\tilde{\phi}^+_{BB'})], \tag{5.7b}$$

$$\chi^{a,b}_{A'B'} := \text{Tr}_{A_I,A'_I,B_I,B'_I}\Big[\tilde{W}^{a,b}(\mathbb{I}_{A'B'}\otimes\tilde{\phi}^+_{A_IA'_I}\otimes\tilde{\phi}^+_{B_IB'_I})\Big] \tag{5.7c}$$

Eq.(5.7a) tells that, any protocol followed by Alice and Bob on the joint process $\mathscr{B}^{\mathbf{x}}_{AB}\otimes W_{A_IA_OB_IB_O}$ boils down to performing a POVM $\{\Pi^{a,b}_{AB}\}^1_{a,b=0}$ on $\mathscr{B}^{\mathbf{x}}_{AB}$. Consider now, $W_{A_IA_OB_IB_O}$ is a PPT (positive partial transpose) operator across $A_IA_O|B_IB_o$ bipartition, *i.e.*, $W^{\text{T}_{B_IB_O}}_{A_IA_OB_IB_O}\geq 0$. Furthermore, N^a and M^b being local CP maps on Alice's and Bob's parts, their corresponding dual maps $N^{(\star)a}$ and $M^{(\star)b}$ are also

[1]Recall that, a map $\Lambda^{(*)}_{D\to C}$ is called dual to $\Lambda_{C\to D}$ if $\text{Tr}_D[\rho_D\{\Lambda_{C\to D}(\sigma_C)\}] = \text{Tr}_C[\{\Lambda^{(*)}_{D\to C}(\rho_D)\}\sigma_C]$ for all $\rho_D\in\mathscr{D}(H_D)$ & $\sigma_C\in\mathscr{D}(H_C)$.

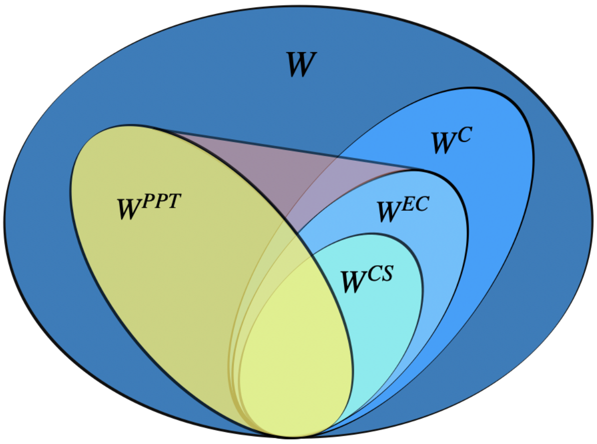

FIG. 5.3 Within the set $\mathbf{W}$ of all bipartite processes, $\mathbf{W}^{PPT}$ is the set of processes that are PPT across $A_IA_O|B_IB_O$ bipartition and $\mathbf{W}^{CS}, \mathbf{W}^{EC}$ and $\mathbf{W}^{C}$ denote the set of causally-separable, extensibly causal and causal processes. Processes lying within the convex hull of $\mathbf{W}^{PPT}$ and $\mathbf{W}^{EC}$ yield $P_{succ}^{\text{DR-B}} \leq 1/2$.

CP on the respective parts. This ensures

$$\left(\tilde{W}^{a,b}_{A_IA'A'_IB_IB'B'_I}\right)^{\mathrm{T}_{B_IB'B'_I}} \geq 0, \ \forall\ a,b \in \{0,1\}. \tag{5.8}$$

Eq.(5.7c) and Eq.(5.8) together imply $\chi^{a,b}_{A'B'} \in \mathrm{PPT}(\mathbb{C}^2_{A'} \otimes \mathbb{C}^2_{B'})$, and in-fact due to Peres-Horodecki criteria [101, 102], $\chi^{a,b}_{A'B'} \in \mathrm{Sep}(\mathbb{C}^2_{A'} \otimes \mathbb{C}^2_{B'})$. Now, Eq.(5.7b) ensures $\Pi^{a,b}_{AB} \in \mathrm{Sep}(\mathbb{C}^2_{A} \otimes \mathbb{C}^2_{B})$. Therefore, Eqs.(5.5a) & (5.7a) imply

$$P_{succ}^{\text{DR-B}} = \frac{1}{4}\sum_{x_1,x_2} \mathrm{Tr}_{A,B}\left[\mathscr{B}^{x_1x_2}_{AB}\Pi^{a=x_1,b=x_2}_{AB}\right], \tag{5.9}$$

which can be thought of as the success probability of distinguishing two-qubit Bell Basis $\{|\phi^{\pm}\rangle, |\psi^{\pm}\rangle\}$ under separable measurement $\{\Pi^{a,b}\}_{ab}$. Recalling the result from [110], we know this success probability is upper bounded by $1/2$. This concludes the proof. □

Note that, Theorem [15] provides only a necessary criterion on bipartite processes to be useful in DR-B task. The fact that it is not a sufficient criterion can be seen from the example of the no-signaling process $W_{A_IA_OB_IB_O} := |\phi^+\rangle_{A_IB_I}\langle\phi^+| \otimes \mathbb{I}_{A_OB_O}$. This particular process is NPT (negative-partial-transpose) across $A_IA_O|B_IB_O$ bipartition, but according to Proposition [6] it does not provide a nontrivial success in DR-B task.

Theorem [15] also indicates that not all causally inseparable processes are advantageous in DR-B task. For instance, consider the process $W^{Cyril}_{A_IA_OB_IB_O}$ in Eq.

(2.73), which is known to be advantageous in GYNI task. However, it turns out that $(W^{Cyril}_{A_IA_OB_IB_O})^{\mathrm{T}_{B_IB_O}} \geq 0$, implying $W^{Cyril}_{A_IA_OB_IB_O}$ to be PPT across $A_IA_O|B_IB_O$ bipartition. Moreover, it admits a fully separable decomposition across $A_I|A_O|B_I|B_O$ partition[2]:

$$\begin{aligned}
W^{Cyril}_{A_IA_OB_IB_O} &= \frac{1}{4}\left[\mathbb{I}^{\otimes 4} + \frac{1}{\sqrt{2}}\left(\sigma^3\sigma^3\sigma^3\mathbb{I} + \sigma^3\mathbb{I}\sigma^1\sigma^1\right)\right]_{A_IA_OB_IB_O} \\
&= \frac{1}{2}\Big[P^z_+P^z_+P^\alpha_+P^x_+ + P^z_-P^z_+P^\alpha_-P^x_+ + P^z_+P^z_+P^\beta_+P^x_- \\
&\quad + P^z_-P^z_+P^\beta_-P^x_- + P^z_+P^z_-P^\beta_-P^x_+ + P^z_-P^z_-P^\beta_+P^x_+ \\
&\quad + P^z_+P^z_-P^\alpha_-P^x_- + P^z_-P^z_-P^\alpha_+P^x_-\Big]_{A_IA_OB_IB_O}\ ; \qquad (5.10) \\
\text{where,}\ \ P^z_\pm &:= \frac{1}{2}\left(\mathbb{I} \pm \sigma^3\right),\quad P^\alpha_\pm := \frac{1}{2}\left(\mathbb{I} \pm \frac{1}{\sqrt{2}}\left(\sigma^3 + \sigma^1\right)\right), \\
P^x_\pm &:= \frac{1}{2}\left(\mathbb{I} \pm \sigma^1\right),\quad P^\beta_\pm := \frac{1}{2}\left(\mathbb{I} \pm \frac{1}{\sqrt{2}}\left(\sigma^3 - \sigma^1\right)\right).
\end{aligned}$$

While Proposition [6] excludes all causally separable bipartite processes to be useful in DR-B task, Theorem [15] excludes processes that are PPT in Alice vs Bob bipartition. In fact, a larger class of bipartite processes can be excluded for the task in question.

Corollary 3. *Any bipartite process matrix W will not provide a nontrivial success in DR-B task if it can be obtained through probabilistic mixture of two other processes W' & W'', where W' is extensibly causal and W'' is PPT in Alice vs Bob bipartition.*

Proof simply follows from Corollary [2] and Theorem [15] due to the fact that success probability is a linear function of the processes. A pictorial depiction of this corollary is shown in Fig.5.3. We now proceed to present an intriguing super-activation phenomenon involving process matrices.

5.5 Super-Activation Phenomenon

Super-activation, where two or more 'useless' resources become 'useful' when combined for a specific task, is a ubiquitous phenomenon in quantum information processing. Here, we demonstrate super-activation in the context of causal indefiniteness. Similar results have been established in previous studies [61, 60], where the violation of certain causal inequalities was used as the utility function to show super-activation. However, super-activation can manifest with different utility functions. Here, we consider the success probability in the DR-B task

[2] A positive operator $\mathscr{O}_{YZ} \in \mathscr{L}_+(H_Y \otimes H_Z)$ is called separable across $Y|Z$ cut if it allows a decomposition of the form $\mathscr{O}_{YZ} = \sum_i \mathscr{O}^i_Y \otimes \mathscr{O}^i_Z$, where $\forall\ i,\ \mathscr{O}^i_Y \in \mathscr{L}_+(H_Y)$ & $\mathscr{O}^i_Z \in \mathscr{L}_+(H_Z)$.

as the utility function and address the question: *Can there be two quantum processes W and W', neither providing any advantage in the DR-B task, but yield a nontrivial success while their composition $W \otimes W'$ is considered?*

Before addressing this question, a careful analysis is required to determine whether the composite object $W \otimes W'$ represents a valid quantum process. As pointed out by Jia & Sakharwade [111], generally $W \otimes W'$ violates the normalization condition of probabilities, leading to paradoxes (see also [112]). However, the composition represents a valid quantum process when one of them is a no-signaling process, namely a bipartite quantum state, and the other is any general quantum process. In fact, the existence of such a composition is required to prove the positivity of a generic quantum process matrix [11]. Particularly, Eq. (2.64) ensures $W_{A_IA_OB_IB_O} \in \mathscr{L}(H_{A_I} \otimes H_{A_O} \otimes H_{B_I} \otimes H_{B_O})$ to be POPT (Positive on Product test) operator, whereas its positivity in ensured in Eq. (2.65), demanding existence of the composite process $W_{A_IA_OB_IB_O} \otimes \rho_{AB}$. Thus, the question of super-activation of causal indefiniteness makes sense, and we provide an affirmative answer to this question.

Theorem 16. *There exists two processes W and W' such that the success probability for DR-B task is bounded by $1/2$ for both the processes, but when used in composition $W \otimes W'$ we can achieve a success strictly greater than $1/2$.*

Proof. The proof is constructive. Considering $W = W^{Cyril}_{A_IA_OB_IB_O}$ and $W' = \phi^+_{AB}$. While Proposition [6] bounds success probability of DR-B task for W' to be $1/2$, Theorem [15] imposes the same bound for W. In both cases, the success $1/2$ can be achieved simply by following the protocol stated in Proposition [3]. On the other hand, using the protocol stated in Eq.(2.73), a success $5/16(1+1/\sqrt{2}) > 1/2$ can be achieved in GYNI game with the process W. Therefore, following the protocol discussed in the '*if part*' proof of Theorem [14], we can obtain the success $5/16(1+1/\sqrt{2})$ in the DR-B task with the composite process $W \otimes W'$. This establishes the super-activation of causal indefiniteness. □

Notably, the pair $(W,W') \equiv (W^{Cyril}, \phi^+)$ is not the only instance of process-pair exhibiting such super-activation phenomenon – here W^{Cyril} can be replaced by any process $W \in \text{ConvHull}(\mathbf{W}^{EC} \cup \mathbf{W}^{PPT}) \setminus \mathbf{W}^{EC}$ and yielding nontrivial advantage in GYNI game (see fig.(5.3) for definition of the sets $\mathbf{W}^{EC}, \mathbf{W}^{PPT}, \mathbf{W}^{EC}$). An interesting question is which other no-signaling processes (*i.e*, bipartite quantum states) can be used as W'_{AB} to activate causal indefiniteness of such $W_{A_IA_OB_IB_O}$'s? A partial answer follows from Theorem [15]. Any PPT state ρ^{PPT}_{AB} cannot be used for the purpose as $W_{A_IA_OB_IB_O} \otimes \rho_{AB}^{PPT}$ is PPT across $AA_IA_O|BB_IB_O$ whenever $W_{A_IA_OB_IB_O}$ is PPT across $A_IA_O|B_IB_O$. In general, it would be of interest to explore which NPT states are useful for this purpose.

At this point, the results in [61, 60] are worth mentioning. In [61], the authors introduce the notion of causal and causally separable quantum processes. While the causal processes never violates any causal inequality, the causally separable processes allow a canonical decomposition (see Theorem [**2.2**] in [61]).[3] The authors also provide example of a tripartite quantum process that is causal but not causally separable. They also show example of tripartite causally separable processes that become non-causal when extended by supplying the parties with entangled ancillas. This exhibits a kind of 'causal activation' phenomenon. In [60], the authors provide example of bipartite causally nonseparable processes that allow causal model, and they also show evidence of 'causal activation' phenomenon where combination of two causal process becomes non-causal.

5.6 Advantage of classical causal-indefinite processes in DR task

Assuming quantum theory to be valid locally, relaxation of global time order between multiple parties led to the formalism of Process Matrices that accommodates the notion of causal indefiniteness [11]. Notably, by assuming local operations to be strictly classical the authors in [11] have shown impossibility of bipartite causally inseparable processes in classical case, conjecturing the same to hold in the multipartite setting as well. However, quite surprisingly the authors in [113, 114] prove the above conjecture to be false, implying causal indefiniteness to be a feature not inherent to quantum theory only. In this section we will analyse whether such causally indefinite classical process could be advantageous in multipartite DR task. Here we present the main findings by considering a specific example of a tripartite DR task and refer to Appendix [**??**] for detailed explanation of the prerequisites and results.

5.6.1 Tripartite DR task (T-DR)

A Referee encodes the strings $\mathbf{x} \equiv \mathbf{x_1 x_2 x_3} \equiv x_1x_1'x_2x_2'x_3x_3' \in \{0,1\}^{\times 6}$ into

$$\rho_{ABC}^{\mathbf{x}} = (\mathscr{B}_{AC}^{\mathbf{x_1}})^{\otimes 2} \otimes (\mathscr{B}_{BA}^{\mathbf{x_2}})^{\otimes 2} \otimes (\mathscr{B}_{CB}^{\mathbf{x_3}})^{\otimes 2}. \tag{5.11}$$

[3] In a sense, they have analogy with the notions of Bell-local and separable quantum states.

and distributes respective subsystems to Alice, Bob, and Charlie. The hiding condition is satisfied as

$$\rho^{\mathbf{x}}_{\mathscr{K}} := \mathrm{Tr}_{\bar{\mathscr{K}}}\, \rho^{\mathbf{x}}_{ABC} = (\mathbb{I}/2)^{\otimes 4}, \quad \forall\, \mathbf{x}, \tag{5.12}$$

for all $\mathscr{K} \in \{A,B,C\}$, where $\bar{A} := BC$ and *etc.* Each player guesses a two bit string and accordingly will be given some payoff. Their guesses are correct if they have some definitive information about the given messages. For instance, Alice's guess a_1a_1' could be correct in two ways: (i) she perfectly predicts the given string x_1x_1', (ii) she perfectly eliminates one of the strings not given to her. Let us define the sets

$$\$^{\mathbf{y}} \equiv \$^{yy'} := \{0y\bar{y}', 0\bar{y}y', 0\bar{y}\bar{y}', 1yy'\}, \tag{5.13a}$$

$$£^{\mathbf{x}} \equiv £^{\mathbf{x_1x_2x_3}} := \$^{\mathbf{x_1}} \times \$^{\mathbf{x_2}} \times \$^{\mathbf{x_3}}. \tag{5.13b}$$

Here $\bar{0} = 1$ and $\bar{1} = 0$. Accordingly, the winning condition reads as

$$\left\{ \begin{array}{c} \left(\mathbf{a} := a_0a_1a_1' \in \$^{\mathbf{x_1}}\right) \wedge \left(\mathbf{b} := b_0b_1b_1' \in \$^{\mathbf{x_2}}\right) \\ \wedge \left(\mathbf{c} := c_0c_1c_1' \in \$^{\mathbf{x_3}}\right) \end{array} \right\},$$

$$\text{or equivalently,} \quad \mathbf{g} \equiv \mathbf{abc} \in £^{\mathbf{x}}. \tag{5.14}$$

The first bit of a player's guess, *i.e.*, $a_0/b_0/c_0$ denotes whether they chooses to identify the string given to them or choose to eliminate it. Our next result shows that success of T-DR is non-trivially bounded for any such bi-causal process however there exists a tripartite classical process for which this bound can be surpassed. This establishes the genuine tripartite indefiniteness of the corresponding classical process.

Theorem 17. *For any bi-causal process the success of T-DR task is always upper bounded by* $3/4$*, however there exists a tripartite classical process that yields a success of* $27/32 > 3/4$.

The proof is provided in the Appendix [C.3.2]. See Prop.[10].

5.7 Discussions

In this work, we study causal indefiniteness with respect to its utility in retrieving locally inaccessible data. We consider a simple data retrieval (DR) task and demonstrate that, under the single-opening setup, parties sharing causally inseparable processes generally outperform those sharing causally separable processes. Along these lines, we present several intriguing findings, which are discussed comprehensively below, along with their nontrivial implications.

• *Duality between the DR task and the GYNI game.*– For the bipartite case, we demonstrate (Theorems [14] and [C.1]) that if two parties achieve a success probability μ in the DR task from maximally entangled states, a dual protocol exists to achieve the same success in the corresponding GYNI game, and vice versa. This implies that the optimal success probabilities for these seemingly distinct tasks are identical. Duality often plays an important role in both mathematics and physics by bridging seemingly distinct concepts, and providing alternative approaches to solving problems that are challenging in one domain but simpler in their dual formulations. Our established duality offers promising insights into the nature of indefinite causal structures. In particular, a key question in causal indefiniteness is determining the optimal violation of causal inequalities by quantum processes (analogous to the Tsirelson bound for quantum nonlocal correlations).

This question has been partially explored in a recent work by Liu and Chiribella [115], which proposes nontrivial upper bounds for general causal inequalities, demonstrating their achievability for specific classes, called the single-trigger inequalities. In this context, the duality we establish could provide an alternative framework for tackling this problem. Specifically, the optimal success probability for the DR task hinges on the maximum information retrievable from bipartite ensembles in a single-opening setup. Hence, deriving a Holevo-like bound for this scenario could directly inform a Tsirelson bound for causal indefiniteness. This represents a compelling direction for future exploration, with potential to deepen our understanding of quantum causal structures.

• *Peres-like criterion for bipartite quantum processes.*– We demonstrate that causal inseparability alone is insufficient to provide a nontrivial advantage in the data retrieval (DR) task. As a result, we derive a stricter necessary criterion for bipartite quantum processes to be useful in the bipartite data retrieval task from Bell states (DR-B). Specifically, we show that bipartite processes $W_{A_IA_OB_IB_O}$ that are positive under partial transpose (PPT) across the $A_IA_O|B_IB_O$ partition are not useful in DR-B (Theorem [15]), even if they violate a causal inequality. Consequently, in Corollary [3], we establish that any bipartite process lying within the convex hull of extensibly causal processes and PPT processes, ConvHull($\mathbf{W}^{EC} \cup \mathbf{W}^{PPT}$) [see Fig. (5.3)], does not provide any nontrivial advantage in the DR-B task. However, it remains an open question whether there exist processes outside the convex hull that still fail to yield nontrivial success in the DR-B task.

This result introduces a further layer of classification in the quantum process space. Since NPT (non-positive under partial transpose) states are generally more resourceful than PPT states in the LOCC paradigm, a natural expectation is that a similar hierarchy should also manifest at the process level. However, to the best of our knowledge, no operational task exhibiting this hierarchy has been identified in the literature. Our DR-B task provides an explicit demonstration of such a task, highlighting the greater resourcefulness of NPT processes compared to PPT ones.

• *Super-activation of causal indefiniteness.–* We have also reported an intriguing super-activation phenomenon involving quantum processes. Particularly, an entangled state shared between Alice and Bob, being a no-signalling resource, by its own, does not provide a nontrivial success in DR-B task. On the other hand, a process lying within the set ConvHull($\mathbf{W}^{EC} \cup \mathbf{W}^{PPT}$) is also not useful for this task by its own. However, as shown in the **only if** part of our Theorem [14], any process $W \in$ ConvHull($\mathbf{W}^{EC} \cup \mathbf{W}^{PPT}$) $\setminus \mathbf{W}^{EC}$ violating GYNI inequality will become useful in DR-B task when assisted with a two-qubit maximally entangled state, demonstrating the super-activation phenomenon. An explicit example of such a process is the W^{Cyril} process of Eq. (2.73).

• *Advantage of causally inseparable classical processes.–* Considering the tripartite version of DR task (T-DR), we have shown that the advantage of causal indefiniteness in DR task is not exclusive to the quantum nature of process matrices, rather it persists in classical processes as well. Our T-DR task demonstrates that certain tripartite classical processes can outperform bicausal quantum processes (Section [C.3]), establishing the efficacy of genuine causal indefiniteness.

To conclude, our exploration sheds new light on several previously unexplored facets of the causal indefiniteness. On one hand, we uncover new structural characterizations of quantum processes; on the other hand, the established duality between the DR task and the GYNI game opens up the possibility of an information-theoretic approach to address the question of the optimal quantum violation of causal inequalities. Establishing similar dualities between generalized versions of DR tasks and other causal games may provide new foundational insights into the structure of causally indefinite processes, paving the way for future research directions.

Chapter 6

Classical simulation of composite system statistics

6.1 Introduction

Classical physics, rooted in intuitive and objective principles, offers deterministic descriptions of the physical phenomena we encounter in daily life. In stark contrast, the quantum realm defies classical reasoning, exhibiting phenomena that challenge conventional intuition. Quantum mechanics—formulated within the Hilbert space framework—delivers an extraordinarily precise mathematical account of these phenomena, but it refrains from offering clear physical intuition about their nature [116–118]. Nonetheless, the advent of quantum information theory has highlighted practical advantages of quantum resources over their classical counterparts in tasks such as computation, communication, and cryptography [19, 20, 119–125]. In this context, simulating quantum processes with classical resources promises a compelling research avenue [126–128]. Such investigations serve a dual purpose: quantifying the computational and communicational power of quantum resources while deepening our understanding of the unique features that distinguish quantum phenomena from classical intuitions.

A hallmark of quantum mechanics, underscored by Bell's theorem [1] and corroborated through decades of experiments [2, 129–134], is the emergence of nonlocal correlations among the outcomes of local measurements performed on entangled states. These correlations defy any *local realistic* explanation [62, 63, 135, 64, 136]. Furthermore, entangled states shared among distant parties cannot be prepared through local quantum operations and classical communication (LOCC) [84]. Despite their inherent nonlocality, the local measurement statistics of entangled states can often be faithfully reproduced through finite classical communication between distant parties holding parts of the composite system [137–145]. This paradigm extends naturally to quantum channel simulation, where a receiver (Bob) aims to replicate the statistics of arbitrary measurements on a quantum state unknown to him but fully known to a sender (Alice), who aids Bob while minimizing the classical communication required [146–150]. In particular, the result by Toner and Bacon demonstrated that the statistics of any projective measurement, also called the von Neumann measurement, on a qubit state can be simulated using just two classical bits of communication [147]. Subsequent work extended this result to more general settings, including positive operator-valued measures (POVMs) [23], further illustrating the feasibility of classical simulation with finite communication [150].

In this chapter, we argue that quantum channel simulation must go beyond replicating local measurement statistics to address more general scenarios. Specifically, simulations must account for the statistics of composite measurements including entangled basis measurements on Alice's system and an ancillary system held by Bob. The generalised simulation scenario is discussed in the next section.

6.2 Generic simulation of a qubit known to sender

The simulation of a quantum channel must perfectly reproduce the statistical distribution of all possible measurement outcomes, as prescribed by the Born rule. This encompasses the broader scenario of channel simulation, wherein the objective is to reproduce the outcome statistics of a joint measurement performed at Bob's end.

To formally define this task, consider a scenario in which Alice is provided with the classical description of a qubit state, given by $\psi := \frac{1}{2}(\mathbf{I}_2 + \hat{\psi}\cdot\vec{\sigma})$, where $\hat{\psi}$ denotes the Bloch vector representation of ψ. Simultaneously, Bob is given another quantum state ϕ (not necessarily a qubit), which remains unknown to both Alice and Bob. If Alice transmits the physical qubit to Bob, it may experience noise characterized by a quantum channel Λ. Consequently, Bob possesses the composite state $\Lambda(\psi_A)\otimes\phi_B$ in his laboratory. He may then choose to implement a joint measurement $\mathbf{M}_{AB} \equiv \{\mathrm{E}^k_{AB}\}$ on the combined system.

According to the Born rule, the probability of obtaining the measurement outcome k associated with the effect E^k_{AB} is given by

$$p(k|\psi_A\otimes\phi_B) = \mathrm{Tr}\left[(\Lambda(\mathrm{P}_\psi)\otimes\mathrm{P}_\phi)\mathrm{E}^k_{AB}\right], \tag{6.1}$$

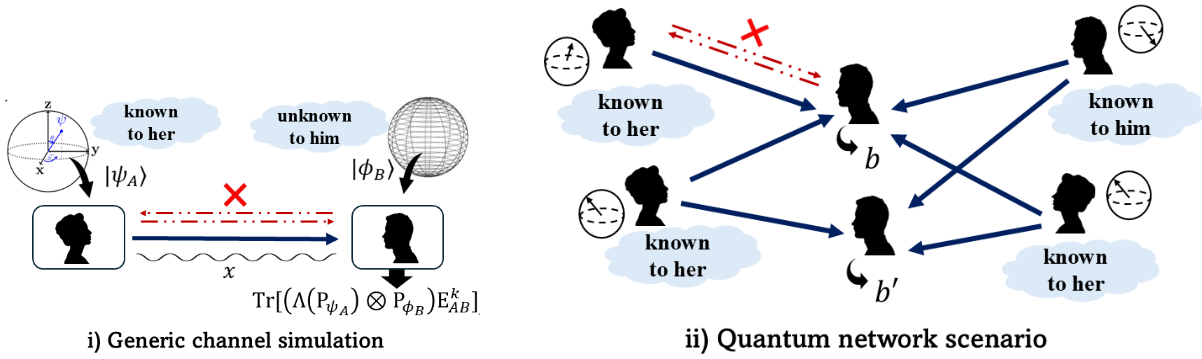

FIG. 6.1 (i) The generic simulation of a qubit channel Λ. Alice is provided with classical description of a qubit state $\psi_A \in \mathbb{C}^2_A$, while Bob holds an unknown state $\phi_B \in \mathbb{C}^2_B$. Their goal is to reproduce the statistics of a joint measurement $\mathrm{M}_{AB} \equiv \{\mathrm{E}^k_{AB}\}$ at Bob's location, which as per Born rule reads as $\mathrm{Tr}[(\Lambda(\mathrm{P}_{\psi_A}) \otimes \mathrm{P}_{\phi_B})\mathrm{E}^k_{AB}]$. More generally, the system B can have arbitrarily large dimension and may also form part of a larger joint system BC. Solid arrow denotes a quantum channel, dashed arrow denotes classical communication line, and wavy line denotes shared classical correlation. Our Theorem 19 establishes a fundamental gap between classical and quantum resources in this setup. (ii) The situation naturally arises in quantum network, where some nodes possess the classical description of the quantum state while others do not.

where P_ψ and P_ϕ denote the projectors onto the respective states (see Fig. [6.1]). Importantly, the state of system B may be unknown to both parties and could even constitute a subsystem of a larger entangled state, such as BC. The current work aims to show that this generalized framework introduces significant challenges for classical simulation protocols. Importantly, the existing channel simulation protocols consider only one-round protocols with communication from Alice to Bob; and communication cost of such a protocol is defined as the minimum communication required for exact simulation [151, 152]. In general, however, one may allow multi-round interactive protocols involving back-and-forth classical communication, which we explore in subsequent sections.

In the following section, we establish a general no-go theorem, demonstrating that classical simulations of such qubit-based protocols are, in general, highly inefficient in terms of the required classical communication cost.

6.3 A no-go theorem for simulation of qubit channel

Theorem 18. *The generic simulation of the perfect qubit channel is impossible through one-round protocol by using classical resources alone, even if Alice is permitted to send an arbitrary but finite amount of classical information to Bob.*

Proof. Here we only provide an outline for the proof. We provide one specific instance where a simulation with finite classical communication is impossible. Consider Bob's state ϕ_B to be a qubit and for now we assume there is no noise in Alice's qubit transmission. Also consider the specific measurement $\mathbf{M}_{\text{singlet}} \equiv \mathbf{M}_{\psi^-} := \{\mathrm{P}_{\psi^-}, \mathbf{I}_4 - \mathrm{P}_{\psi^-}\}$ only; where $|\psi^-\rangle := \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ is the singlet state.The probability of obtaining the outcome $|\psi^-\rangle$ is given by:

$$p_{\psi,\phi} := p(\psi^-|\psi,\phi,\mathbf{M}_{\psi^-}) = \frac{1}{4}(1 - \hat{\psi}\cdot\hat{\phi}). \tag{6.2}$$

The most general one round classical communication protocol that Alice and Bob can implement to simulate the statistics in Eq. (6.2) proceeds as follows: Alice generates a classical variable m, sampled according to the conditional distribution $p(m|x,\psi)$, where x is a shared random variable sampled as $p(x)$, and ψ is the state given to Alice, uniformly sampled from the Bloch sphere; Alice communicates m to Bob; Bob performs a two-outcome POVM $\mathbf{M}^{m,x} \equiv \{\mathrm{E}^{m,x}, \mathbf{I}_2 - \mathrm{E}^{m,x}\}$ on the unknown state ϕ. Since ϕ is unknown to both parties, their protocols are independent of ϕ. Associating $\mathrm{E}^{m,x}$ with the ψ^- outcome in Eq. (6.2), perfect simulation demands:

$$p_{\psi,\phi} = \sum_m \int dx\ p(x)\ p(m|x,\psi)\ \langle\phi|\mathrm{E}^{m,x}|\phi\rangle = \langle\phi|\mathrm{F}_\psi|\phi\rangle,$$

where $\mathrm{F}_\psi := \sum_m \int dx\ p(x)\ p(m|x,\psi)\ \mathrm{E}^{m,x}$ is the effective POVM element implemented by Bob on ϕ, given that Alice is provided with ψ.

Consider now the case where $\phi = \psi$, leading to $p_{\psi,\psi} = 0$, which implies $\mathrm{F}_\psi = \beta_\psi \mathrm{P}_{\hat{\psi}^\perp}$, with $\beta_\psi \geq 0$ for all ψ; here $|\psi^\perp\rangle$ denotes the state orthonormal to $|\psi\rangle$, with $\hat{\psi}^\perp = -\hat{\psi}$. Next, consider the case where $\phi = \psi^\perp$; here, $p_{\psi,\psi^\perp} = \frac{1}{2}$, implying $\beta_\psi = \frac{1}{2}$ for all ψ. Thus, whenever Alice is given the state ψ, Bob's effective POVM on ϕ must take the form $\left\{\frac{1}{2}\mathrm{P}_{\hat{\psi}^\perp}, \mathbf{I}_2 - \frac{1}{2}\mathrm{P}_{\hat{\psi}^\perp}\right\}$. Since ψ is uniformly sampled from the Bloch sphere and is unknown to Bob, it is straight forward to see that no classical protocol involving only a finite amount of classical communication from Alice to Bob can achieve the desired outcome. A formal proof is listed in the Appendix [D.1] □

Any simulation that uses only a finite amount of classical communication, whether one-way or interactive, remains inherently classical. Extending our no-go theorem to general multi-round protocols therefore sharpens our understanding of the classical limits on emulating quantum dynamics. It is worth noting that multi-round protocols involving bidirectional communication are shown to advantageous both in classical and quantum realms. For instance, communication from receiver to sender (called the feedback assistance) can enhence zero-error capacity of noisy classical channels [153], where entanglement purification protocol under two-way protocol can surpass the one-way optimal bound [154]. It is also known that characterizing the multi-round local quantum operation and classical communication protocols is a hard problem [107]. Given these subtleties, generalizing Theorem 18 beyond single-round communication is nontrivial. Nonetheless, We now show that even the most powerful finite, multi-round classical protocol cannot perfectly simulate a qubit channel. Specifically, we allow arbitrarily large but finite number of back-and-forth rounds between Alice and Bob, each carrying an arbitrarily large but finite amount of classical data, and refer to such schemes as finite back-and-forth classical communication protocols. In the following proposition we show that any protocol involving finite back and forth communication can be implemented by one way communication from Alice to Bob.

Proposition 7. *Any simulation protocol involving finite back-and-forth classical communication can always be implemented with finite classical communication from Alice to Bob, only.*

Proof. (Outline) To establish the no-go result for multi-round case, we begin by noting key structural features of such protocols. Any protocol starting with Bob can be reformulated to begin with Alice by inserting a trivial first message from Alice to Bob. Likewise, since the simulation outcome must be produced at Bob's end, the final round must involve communication from Alice; any final message from Bob is irrelevant. Thus, without loss of generality, any valid protocol consists of an odd number of rounds, beginning and ending with communication from Alice to Bob. With a finite number of rounds and finite communication per round, the protocol can be represented as a finite tree: each path from root to leaf corresponds to a specific sequence of exchanged messages, $(m_1, m_2, \cdots, m_k)$, with branching determined by Alice's input and the shared randomness.

The central idea is that, given this finite structure, Alice can locally compute—in advance—the exact message sequence that would be followed for any input state and shared variable. She can then compress this sequence into a single classical message and send it to Bob, thereby collapsing the multi-round protocol into an equivalent one-round protocol. In Appendix [D.2], we make this construction explicit for a three-round protocol and show that the resulting correlations can be exactly reproduced by a single-round protocol. By iterating this reduction, any finite-round protocol with bounded communication per round can be simulated by a one-round protocol with finite classical communication from Alice to Bob. □

Unless stated otherwise, all subsequent results hold for finite back-and-forth classical protocols, and invoking Proposition [7], we can restrict to one-round protocols only. As an immediate consequence of Proposition [7], the no-go result of Theorem [18] generalizes as follows:

Theorem 19. *The generic simulation of the perfect qubit channel is not possible by any finite back-and-forth classical protocol.*

In communication complexity, one quantifies the quantum advantage by the extra classical communication—typically supplemented by preshared randomness—needed to reproduce quantum statistics. Prior results [146–150] show that, if Bob receives no quantum state, a finite amount of classical communication can successfully simulate a quantum channel. At first glance, this suggests that quantum–classical gaps might be closed with bounded classical resources. Our Theorem [19], however, demonstrates that even in the simplest nontrivial setting—two-dimensional quantum systems no finite back-and-forth classical protocol can emulate a perfect qubit channel.

At this point a fundamental question arises as to whether the necessity of entangled basis measurements is intrinsic to establishing the no-go result in Theorem [18]. More precisely, if the joint measurement comprises solely product or separable effects, can its statistical outcomes be simulated with finite classical communication? We answer this question in the following section.

6.4 A class of composite measurements that are simulable with finite classical communication

Let's start with the simple computational basis measurement:

$$\mathbf{M}_{\text{comp}} \equiv \left\{ \mathrm{P}_{\hat{z}} \otimes \mathrm{P}_{\hat{z}},\ \mathrm{P}_{\hat{z}} \otimes \mathrm{P}_{\hat{z}^{\perp}},\ \mathrm{P}_{\hat{z}^{\perp}} \otimes \mathrm{P}_{\hat{z}},\ \mathrm{P}_{\hat{z}^{\perp}} \otimes \mathrm{P}_{\hat{z}^{\perp}} \right\}, \tag{6.3}$$

can be simulated using only a single bit of classical communication from Alice to Bob. Specifically, Alice measures her state in the σ_z basis and transmits her measurement outcome to Bob, who subsequently measures his own state in the σ_z basis.

A similar protocol applies to the twisted measurement

$$\mathbf{M}^{(B)}_{\text{twist}} \equiv \left\{ \mathrm{P}_{\hat{z}} \otimes \mathrm{P}_{\hat{z}},\ \mathrm{P}_{\hat{z}} \otimes \mathrm{P}_{\hat{z}^{\perp}},\ \mathrm{P}_{\hat{z}^{\perp}} \otimes \mathrm{P}_{\hat{x}},\ \mathrm{P}_{\hat{z}^{\perp}} \otimes \mathrm{P}_{\hat{x}^{\perp}} \right\}, \tag{6.4}$$

where Alice measures her state in the σ_z basis and communicates the result to Bob. Based on Alice's communication, Bob performs either a σ_z or σ_x measurement on his qubit.

The simulation process becomes slightly more intricate when the twist is applied to Alice's side, as in the measurement

$$\mathbf{M}^{(A)}_{\text{twist}} \equiv \left\{ \mathrm{P}_{\hat{z}} \otimes \mathrm{P}_{\hat{z}},\ \mathrm{P}_{\hat{z}^{\perp}} \otimes \mathrm{P}_{\hat{z}},\ \mathrm{P}_{\hat{x}} \otimes \mathrm{P}_{\hat{z}^{\perp}},\ \mathrm{P}_{\hat{x}^{\perp}} \otimes \mathrm{P}_{\hat{z}^{\perp}} \right\}. \tag{6.5}$$

In this scenario, Alice measures her state in both the σ_z and σ_x bases and transmits the outcomes to Bob via two separate 1-bit classical channels. This is feasible as Alice has knowledge of her state and can therefore effectively duplicate it. Bob then measures his qubit in the σ_z basis and selects the appropriate bit from Alice's communication based on his measurement result. An alternate protocol is possible using 1-bit of communication from Bob to Alice, followed by 1-bit from Alice to Bob: Bob performs σ_z measurement and communicates the outcome to Alice, who accordingly performs either σ_z or σ_x and communicates back her outcome. Notably,both protocols involve 2-bit of communication. In this regard the following observation is noteworthy.

Observation 1. *Outcome statistics of* $\mathrm{M}^{(A)}_{twist}$ *on Alice's known qubit and Bob's unknown qubit cannot be reproduce at Bob's laboratory with 1 bit of communication.*

The proof is obtained utilizing the fact that in random-access-code (RAC) task, a qubit provides an advantage over the classical bit [42, 155] (see Appendix [D.3]). Considering the most general product von Neumann measurements, we establish the following result.

Theorem 20. *Statistics of any product von Neumann measurement on a qubit, known to Alice, and an unknown qudit held by Bob can always be simulated at Bob's end by finite classical communication from Alice to Bob.*

Proof. It is known that any product von Neumann measurement in $\mathbb{C}^2 \otimes \mathbb{C}^d$ are implementable under LOCC [156]. Proof of our theorem follows a similar reasoning as of theirs. A generic orthonormal product Basis (OPB) of $\mathbb{C}^2 \otimes \mathbb{C}^d$

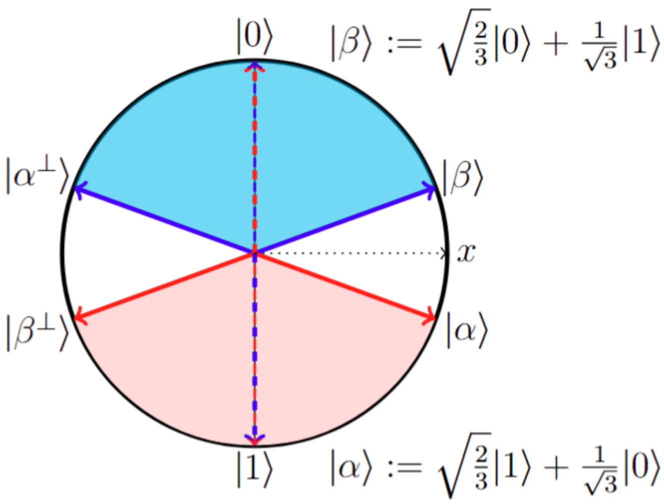

FIG. 6.2 Twisted-butterfly measurement M_{tb}: In Alice's part the projectors $\{\mathrm{P}_{\hat{z}},\mathrm{P}_{\hat{z}^{\perp}},\mathrm{P}_{\hat{\beta}},\mathrm{P}_{\hat{\alpha}^{\perp}}\}$ are involved, while in Bob parts the projectors $\{\mathrm{P}_{\hat{z}},\mathrm{P}_{\hat{z}^{\perp}},\mathrm{P}_{\hat{\alpha}},\mathrm{P}_{\hat{\beta}^{\perp}}\}$ are used.

takes the form $\mathbf{B} = \cup_i \mathbf{B}_i$, with

$$\mathbf{B}_i := \{|\alpha_i\rangle \otimes |\beta_{ij}\rangle, |\alpha_i^{\perp}\rangle \otimes |\tilde{\beta}_{ij}\rangle\}, \tag{6.6}$$

where, $\langle\beta_{ij}|\beta_{i'j'}\rangle = \langle\tilde{\beta}_{ij}|\tilde{\beta}_{i'j'}\rangle = \delta_{ii'}\delta_{jj'}$ and for $i \neq i'$, $\langle\beta_{ij}|\tilde{\beta}_{i'j'}\rangle = \delta_{jj'}$. Notably, the subspaces $\mathbf{S}_i = \mathrm{Span}\{|\beta\rangle_{ij}, |\tilde{\beta}_{ij}\rangle\}_j$ at Bob's part are mutually orthogonal. To simulate the statistic of von Neumann measurement on the basis $\mathbf{B}$ they apply the following protocol: (i) Bob performs a measurement distinguishing the subspaces $\mathbf{S}_i$'s, while Alice performs measurements $\mathrm{M}_i \equiv \{\mathrm{P}_{\alpha_i},\mathrm{P}_{\alpha_i^{\perp}}\}$ on different copies of her known state, and through different classical channels she communicates $0_i(1_i)$ whenever the projector $\mathrm{P}_{\alpha_i}(\mathrm{P}^{\perp}_{\alpha_i})$ clicks; (ii) Bob considers the communication from i^{th} channel if his projector corresponding to $\mathbf{S}_i$ subspace clicks, and then he performs a measurement that distinguishes the states $\{|\beta\rangle_{ij}\}$ if Alice's communication is 0_i, otherwise he performs a measurement that distinguishes the states $\{|\tilde{\beta}\rangle_{ij}\}$. This completes the proof. An explicit example is discussed in Appendix [D.4]. □

Theorem [20], however, does not fully resolve the question of whether all product POVMs can be simulated with a finite amount of classical communication from Alice to Bob, as there exist measurements involving only rank-1 product effects, but not LOCC implementable. For instance, inspired by an construction in [157], we consider the following POVM:

$$\mathrm{M}_{tb} \equiv \left\{ \begin{array}{c} \Pi_1 := \mathrm{P}_{\hat{z}} \otimes \mathrm{P}_{\hat{z}^\perp} \\ \Pi_{21} := \kappa\ \mathrm{P}_{\hat{\alpha}^\perp} \otimes \mathrm{P}_{\hat{z}}, \Pi_{22} := \kappa\ \mathrm{P}_{\hat{z}^\perp} \otimes \mathrm{P}_{\hat{\alpha}} \\ \Pi_{31} := \kappa\ \mathrm{P}_{\hat{\beta}} \otimes \mathrm{P}_{\hat{z}}, \Pi_{32} := \kappa\ \mathrm{P}_{\hat{z}^\perp} \otimes \mathrm{P}_{\hat{\beta}^\perp} \end{array} \right\}, \tag{6.7}$$

where $\kappa := 3/4$. We call this the *twisted-butterfly* POVM M_{tb}, a name justified by its structure (see Fig.[6.2]).

Lemma 3. *The POVM* M_{tb} *is not implementable by Alice and Bob under the operational paradigm of LOCC.*

Proof. The proof simply follows an argument provided in [157]. The measurement M_{tb} perfectly distinguishes the set of orthonormal states $S_3 \equiv \{|\psi_1\rangle := |01\rangle,\ |\psi_2\rangle := (|\phi^-\rangle - |10\rangle)/\sqrt{2},\ |\psi_3\rangle := (|\phi^-\rangle + |10\rangle)/\sqrt{2}\} \subset \mathbb{C}^2 \otimes \mathbb{C}^2$, as $\mathrm{Tr}[\Pi_1 \mathrm{P}_{\psi_j}] = \delta_{1j}$ and $\mathrm{Tr}[(\Pi_{i1} + \Pi_{i2})\mathrm{P}_{\psi_j}] = \delta_{ij}$, for $i \in \{2,3\}$ & $j \in \{1,2,3\}$; here $|\phi^-\rangle := (|00\rangle - |11\rangle)/\sqrt{2}$. On the other hand, $|\psi_2\rangle$ & $|\psi_3\rangle$ being entangled, the set S_3 is LOCC indistinguishable [158]; and hence proves the claim. □

Although the measurement M_{tb} is not LOCC implementable, quite interestingly, it turns out that the statistics of this measurement on a qubit state known to Alice and an unknown qubit state provided to Bob can be simulated at Bob's end with finite classical communication from Alice (See Appendix [D.5]). Instead of proving this particular claim, in the following we establish a more generic result.

Theorem 21. *Statistics of any separable measurement on a quantum state known to Alice and an unknown state of another quantum system provided to Bob, can always be simulated at Bob's end by finite classical communication from Alice to Bob.*

The proof is provided in Appendix [D.6]). Here, we note that Theorem [21] admits a natural generalization to multipartite settings. In this scenario, multiple distant senders—Alice-1, Alice-2, $\cdots$, Alice-n—each receive a classical description of a local quantum state $\psi_{A_i} \in \mathbb{C}^{d_i}_{A_i}$, known only to the i^{th} sender. The receiver, Bob, holds an unknown state $\phi_B \in \mathbb{C}^d_B$ and aims to reproduce the statistics of a K-outcome measurement $\mathrm{M}_{A_1\cdots A_n B} \equiv \{\Pi^b_{A\_1\cdots A_n B} \mid b = 1, \cdots, k\}$ on the joint state $\otimes_{i=1}^n \psi_{A_i} \otimes \phi_B$. As shown in Theorems [24] and [25] (see Appendix [D.7]), the statistics can be simulated using only finite classical communication among the parties whenever each of the effects $\Pi^b_{A_1\cdots A_n B}$'s are fully separable [84].

Theorem [21] is important as it establishes that the no-go result in Theorem [18] necessitates considering a measurement involving entangled effects on the joint system of Alice and Bob.

In the following section we study the scenario of simulating a noisy qubit communication.

6.5 Generic simulation of a noisy qubit

Thus far, we have focused on the simulation of perfect qubit channel. A natural extension is to ask whether the no-go result of Theorem [18] applies to imperfect qubit channels. To address this, we consider the qubit depolarizing channel $\mathrm{D}_\eta : \mathscr{D}(\mathbb{C}^2) \to \mathscr{D}(\mathbb{C}^2)$, defined as $\mathrm{D}_\eta(\rho) := \eta\rho + (1-\eta)\frac{\mathbf{I}_2}{2}$, where $\eta \in [0,1]$. We now analyze the classical simulability of this particular class of channels.

Theorem 22. *For all $\eta \in [0,1)$, the qubit depolarizing channel D_η can be simulated with a finite amount of classical communication from Alice to Bob. The required communication increases as $\eta \to 1$.*

Proof. Given a known state $\psi = \frac{1}{2}(\mathbf{I}_2 + \hat{\psi}\cdot\vec{\sigma})$, if Alice can ensure that the state $\mathrm{D}_\eta(\psi) = \frac{1}{2}(\mathbf{I}_2 + \eta\ \hat{\psi}\cdot\vec{\sigma})$ is reproduced at Bob's laboratory, then any generic measurement statistics can also be reproduced by Bob. Let Alice be allowed to communicate m classical bits to Bob. To reproduced the state $\mathrm{D}_\eta(\psi)$ at Bob's end their protocol proceeds as follows:-

- (i) Shared Randomness: Alice and Bob share a classical random variable $\mathbf{X} \in \mathrm{U}(\mathbb{C}^2)$, which is drawn Haar-randomly from the set of unitary operators on $\mathbb{C}^2$.
- (ii) Predefined States: Before the protocol begins, Alice and Bob agree on a set of 2^m equally spaced Bloch vectors $\{\hat{\omega}_i\}_{i=1}^{2^m}$ with the corresponding qubit states $\{\omega_i\}_{i=1}^{2^m}$.
- (iii) Overlap Computation and Communication: Given the input state ψ, Alice computes the overlaps $\mathrm{Tr}[\mathbf{X}\mathrm{P}_{\hat{\omega}_i}\mathbf{X}^\dagger\mathrm{P}_{\hat{\psi}}]$ for all i and identifies the index $i^\star$ that maximizes this overlap. She communicates the index $i^\star$ to Bob using m-bit classical communication.
- (iv) State Preparation at Bob's End: Upon receiving $i^\star$ and having access to the shared variable $\mathbf{X}$, Bob prepares the state $\mathbf{X}\mathrm{P}_{\hat{\omega}_{i^\star}}\mathbf{X}^\dagger$.

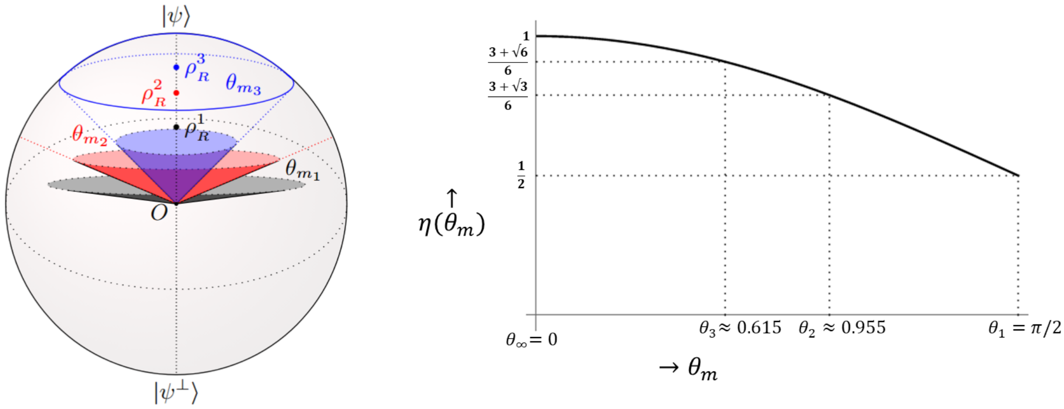

FIG. 6.3 [Left] Generic Simulation of the Qubit Depolarizing Channel $\mathrm{D}_\eta : \mathscr{D}(\mathbb{C}^2) \rightarrow \mathscr{D}(\mathbb{C}^2)$. Given a qubit state $\psi = \frac{1}{2}(\mathbf{I}_2 + \hat{\psi}\cdot\sigma)$ to Alice, the state prepared at Bob's end is of the form $\mathbf{X}\mathrm{P}_{\hat{\omega}_{i^\star}}\mathbf{X}^\dagger$, where $\mathbf{X}$ is a Haar-random unitary on $\mathbb{C}^2$, and the vector $\hat{\omega}_{i^\star}$ lies within a cone of half apex angle θ_m centered around $\hat{\psi}$. Averaging over the random unitaries $\mathbf{X}$, the state at Bob's end becomes $\frac{1}{2}(\mathbf{I}_2 + \eta(\theta_m)\,\hat{\psi}\cdot\sigma)$, effectively simulating a depolarizing channel with parameter $\eta(\theta_m)$. As the communication m increases, the apex angle θ_m decreases, leading to a higher value of $\eta(\theta_m)$, and hence a less noisy depolarizing channel. Here we illustrate three such cases (not to scale), shown respectively in black, red, and blue, corresponding to increasing communication levels $m_1 < m_2 < m_3$. [Right] Solid curve depicts variation of $\eta(\theta_m)$ with θ_m [see Eq.(D.36)]. Values of $(\theta_m, \eta(\theta_m))$ for 1-bit, 2-bit, and 3-bit communication are shown.

As shown in the Appendix [D.8], on average the state $\mathrm{D}_\eta(\psi)$ is prepared at Bob's laboratory. The parameter η approaches unity as the number of bits m increases, thus allowing increasingly accurate simulation of the depolarizing channel (see Fig (6.3)). □

Let $\eta(m)$ denote the value of the parameter η achieved following the above m-bit protocol . In general, deriving an exact expression for $\eta(m)$ for arbitrary m is challenging, as it depends on the specific choices of Bloch vectors $\{\hat{\omega}_i\}_{i=1}^{2^m}$ (see Appendix [D.8] for more details). However, for small m's we can have some natural choices of Bloch vectors – (m=1): 2 diametrically opposite vectors, yielding $\eta(1) = 1/2$, (m=2): 4 vectors forming a regular tetrahedron, yielding $\eta(2) = (3+\sqrt{3})/6 \approx 0.789$, and ($m$=3): 8 vectors forming the vertices of a cube, yielding $\eta(3) = (3+\sqrt{6})/6 \approx 0.908$.

6.6 Discussions

We have generalized the channel simulation task which has a long history in literature [146–150]. While the standard simulation scenario allows efficient classical protocols, in the generalized task we have shown that simulation of a perfect qubit channel requires an unbounded amount of classical communication from Alice to Bob, even when augmented with arbitrary pre-shared classical correlation. In particular, even though Alice has complete classical knowledge of her qubit state, the unknown state provided to Bob prohibits an efficient classical simulation.

This finding raises some deep foundational questions. For instance, in the standard simulation scenario, it has been shown that simulating any POVM at Bob's end with finite classical communication from Alice is possible if and only if there exists a ψ-epistemic model underlying quantum theory, where quantum wavefunctions represent an agent's knowledge about the system [149]. Extension of this result to the generalized simulation scenario along with our no-go results (Theorems [19]) would suggest a ψ-ontic nature of the qubit wavefunction. That is, wavefunctions correspond to intrinsic properties of the system rather than merely an observer's knowledge. Such a conclusion would align with the claims of the Pusey-Barrett-Rudolph (PBR) theorem [159]. Additionally, it would offer a pathway to weaken the Preparation Independence assumption used in the PBR theorem, an assumption that has faced criticism [160]. On the other hand, inspired by studies like [161], it would be intriguing to examine the status of Theorem [18] when Bob's unknown state is restricted to a predefined set.

Chapter 7

Summary and Future outlook

- **Chapter 3:** In this chapter, we investigate how different composition rules between quantum systems influence the timelike correlations observed in the resulting composite theory. In particular, we demonstrate that the SEP composition of two elementary qubits can yield strong timelike correlations that are unattainable under the standard quantum composition. Furthermore, we show that this gap in achievable correlations can be made arbitrarily large when considering the SEP composition of a large number of qubits. An interesting future research direction is to identify even stronger timelike correlations arising from non-quantum composition rules and to formulate an information-theoretic principle that constrains the composition rule uniquely to the quantum case. Such a principle would provide a fundamental, information-theoretic justification for the tensor product postulate in quantum theory.

- **Chapter 4:** This chapter presents a systematic framework for characterizing pure entangled states in bipartite polygon theories. We show that for bipartite polygon systems with five or more sides, one obtains the notion of non-maximally entangled states, which are absent in the well-studied bipartite four-sided polygonal state spaces. We also observe a distinctive feature of these polygonal theories: mixed entangled states can exhibit Hardy's paradox, in contrast to two-qubit entangled states in quantum theory. We further establish that, in general GPTs, mixed states cannot exhibit Hardy's paradox due to a key property—prepare-measure reciprocity—present in quantum theory. This study raises broader questions about the fundamental features of quantum theory that determine the set of quantum correlations achievable in Bell scenarios. A deeper exploration of local state-space structures and entanglement properties in such theories could provide further insights into the elusive characterization of quantum correlations.

- **Chapter 5:** In this work, we introduce and analyze a multipartite information-theoretic task—*Data Retrieval* from quantum states. We show that agents operating within an *indefinite causal order* generally outperform those constrained to a definite causal order. For the bipartite version of the Data Retrieval task, defined using Bell states, we establish that the optimal success probability coincides with that of the well-studied causal inequality known as the GYNI game. We also prove that processes that are positive under partial transpose (PPT) with respect to a given bipartition do not outperform causally separable processes, thereby suggesting a new form of resource classification within the process-matrix framework. Furthermore, we propose a tripartite version of the task and demonstrate that even classical indefinite-order processes can offer an advantage. Future work could focus on identifying richer variants of the Data Retrieval task, potentially connected to other causal inequalities, and exploring additional information-theoretic tasks where indefinite causal order provides a strict advantage.

- **Chapter 6:** This chapter examines the simulation of quantum channels in general network scenarios involving composite measurements. We prove that it is, in general, impossible to simulate a given channel in the network using only finite classical communication. Remarkably, this impossibility persists even when arbitrary but finite rounds of two-way communication between sender and receiver are allowed. However, for a specific class of composite measurements—separable measurements—we show the existence of an efficient simulation protocol. Our results also have foundational implications: in standard simulation scenarios, it is known that reproducing the statistics of any POVM at Bob's end via finite classical communication from Alice is possible *if and only if* there exists a ψ-epistemic model of quantum theory, wherein the wavefunction represents an agent's knowledge of an underlying reality. Extending this reasoning to our generalized simulation framework, our findings suggest the impossibility of such ψ-epistemic accounts, instead supporting a ψ-ontic interpretation of the quantum state, where the wavefunction encodes intrinsic physical properties of the system.

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Appendix A

Supplementary Material for Chapter [3]

A.1 Proof of Theorem [6]

Proof. We have already argued in theorem [6] that four qubits are necessary to win $\mathscr{P}_D^{[12]}$ perfectly. Here we show a detailed argument for the thorem, that two SEP-bits suffice for winning this game. Alice starts by choosing the following encoding states

$$\Omega[12] := \begin{Bmatrix} \omega_{zz} = |zz\rangle\langle zz|, & \omega_{xx} = |xx\rangle\langle xx|, & \omega_{yy} = |yy\rangle\langle yy| \\ \omega_{z\bar{z}} = |z\bar{z}\rangle\langle z\bar{z}|, & \omega_{x\bar{x}} = |x\bar{x}\rangle\langle x\bar{x}|, & \omega_{y\bar{y}} = |y\bar{y}\rangle\langle y\bar{y}| \\ \omega_{\bar{z}z} = |\bar{z}z\rangle\langle \bar{z}z|, & \omega_{\bar{x}x} = |\bar{x}x\rangle\langle \bar{x}x|, & \omega_{\bar{y}y} = |\bar{y}y\rangle\langle \bar{y}y| \\ \omega_{\bar{z}\bar{z}} = |\bar{z}\bar{z}\rangle\langle \bar{z}\bar{z}|, & \omega_{\bar{x}\bar{x}} = |\bar{x}\bar{x}\rangle\langle \bar{x}\bar{x}|, & \omega_{\bar{y}\bar{y}} = |\bar{y}\bar{y}\rangle\langle \bar{y}\bar{y}| \end{Bmatrix} \subset \left(V^{SEP}_{\mathbb{C}^2\otimes\mathbb{C}^2}\right)_+, \quad \text{(A.1)}$$

where $|\alpha\beta\rangle := |\alpha\rangle\otimes|\beta\rangle$ and $|\kappa\rangle$ $(|\bar{\kappa}\rangle)$ is the eigenstate of Pauli operator σ_κ with eigenvalue $+1$ (-1), with $\kappa\in\{x,y,z\}$. If two states are orthogonal then they can be perfectly distinguished in quantum theory and hence can also be distinguished in *SEP* theory since all the quantum effects are also allowed in *SEP* theory. However, in the above set there are states that are not orthogonal to each other. For winning the game $\mathscr{P}_D^{[12]}$ in *SEP* theory we need to show that those states are perfectly distinguishable in *SEP* theory. This immediately follows from the result of Arai *et al.* [69]. For completeness of our proof, here we argued the same and provide the detailed calculations. Consider the following two operators mentioned in Eq.(3.2)

$$E_1 = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \end{pmatrix} \quad \& \quad E_2 = \begin{pmatrix} 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 1 \end{pmatrix}. \tag{A.2}$$

Consider an arbitrary bi-partite product state $\omega_{nm} = \frac{1}{2}[\mathbb{I} + \vec{n}.\vec{\sigma}] \otimes \frac{1}{2}[\mathbb{I} + \vec{m}.\vec{\sigma}]$ where $\vec{n} \equiv (n_1, n_2, n_3)$ and $\vec{m} \equiv (m_1, m_2, m_3)$ are the Bloch vectors of the corresponding systems. Straightforward calculation yields

$$\mathrm{Tr}(E_1\omega_{nm}) = \frac{1 + n_1m_1 - n_3m_3}{2} \;\; \& \;\; \mathrm{Tr}(E_2\omega_{nm}) = \frac{1 - n_1m_1 + n_3m_3}{2}. \tag{A.3}$$

For arbitrary choices of Bloch vectors $\vec{n}$ & $\vec{m}$ the above expressions are always positive which assures that $E_1, E_2 \in \mathscr{E}^{SEP}$. Furthermore the fact $E_1 + E_2 = \mathbf{1}$ ensures that $\mathbf{M} \equiv \{E_1, E_2\}$ is a valid measurement is *SEP* theory. From the expression in Eq.(A.3) it is immediate that

$$\begin{aligned} \mathrm{Tr}(E_1\omega_{xx}) &= \mathrm{Tr}(E_1 |xx\rangle\langle xx|) = 1, \quad \mathrm{Tr}(E_1\omega_{zz}) = \mathrm{Tr}(E_1 |zz\rangle\langle zz|) = 0, \\ \mathrm{Tr}(E_2\omega_{xx}) &= \mathrm{Tr}(E_2 |xx\rangle\langle xx|) = 0, \quad \mathrm{Tr}(E_2\omega_{zz}) = \mathrm{Tr}(E_2 |zz\rangle\langle zz|) = 1. \end{aligned}$$

Thus the pair of (nonorthogonal) states $\{\omega_{xx}, \omega_{zz}\}$ can be distinguished perfectly in *SEP* theory with the measurement $\mathbf{M} \equiv \{E_1, E_2\}$.

We will now argue that similar measurements can be constructed for any pair of non-orthogonal states in the set $\Omega[12]$. For that we first note down the following observation.

Observation 2. *For any $E \in \mathscr{E}^{SEP}$ we have $(U_A \otimes U_B)E(U_A^\dagger \otimes U_B^\dagger) \in \mathscr{E}^{SEP}$ where U_A is a unitary on H_A and U_B is a unitary on H_B.*

Consider the following set of unitary operations acting on $\mathbb{C}^2$

$$\mathbf{U} \equiv \left\{ \begin{array}{l} A_0^y := \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}, A_0^z := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, B_0^x := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, B_0^y := \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} \\ A_1^y := \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, A_1^z := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, B_1^x := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, B_1^y := \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \end{array} \right\}. \tag{A.4}$$

	ω_{yy}	$\omega_{y\bar{y}}$	$\omega_{\bar{y}y}$	$\omega_{\bar{y}\bar{y}}$	ω_{zz}	$\omega_{z\bar{z}}$	$\omega_{\bar{z}z}$	$\omega_{\bar{z}\bar{z}}$
ω_{xx}	$B_0^xA_0^y \otimes B_0^xA_0^y$	$B_0^xA_0^y \otimes B_0^xA_1^y$	$B_0^xA_1^y \otimes B_0^xA_0^y$	$B_0^xA_1^y \otimes B_0^xA_1^y$	$B_0^xA_0^z \otimes B_0^xA_0^z$	$B_0^xA_0^z \otimes B_1^xA_1^z$	$B_1^xA_1^z \otimes B_0^xA_0^z$	$B_1^xA_1^z \otimes B_1^xA_1^z$
$\omega_{x\bar{x}}$	$B_0^xA_0^y \otimes B_1^xA_0^y$	$B_0^xA_0^y \otimes B_1^xA_1^y$	$B_0^xA_1^y \otimes B_1^xA_0^y$	$B_0^xA_1^y \otimes B_1^xA_1^y$	$B_0^xA_0^z \otimes B_1^xA_0^z$	$B_0^xA_0^z \otimes B_0^xA_1^z$	$B_1^xA_1^z \otimes B_1^xA_0^z$	$B_1^xA_1^z \otimes B_0^xA_1^z$
$\omega_{\bar{x}x}$	$B_1^xA_0^y \otimes B_0^xA_0^y$	$B_1^xA_0^y \otimes B_0^xA_1^y$	$B_1^xA_1^y \otimes B_0^xA_0^y$	$B_1^xA_1^y \otimes B_0^xA_1^y$	$B_1^xA_0^z \otimes B_0^xA_0^z$	$B_1^xA_0^z \otimes B_1^xA_1^z$	$B_0^xA_1^z \otimes B_0^xA_0^z$	$B_0^xA_1^z \otimes B_1^xA_1^z$
$\omega_{\bar{x}\bar{x}}$	$B_1^xA_0^y \otimes B_1^xA_0^y$	$B_1^xA_0^y \otimes B_1^xA_1^y$	$B_1^xA_1^y \otimes B_1^xA_0^y$	$B_1^xA_1^y \otimes B_1^xA_1^y$	$B_1^xA_0^z \otimes B_1^xA_0^z$	$B_1^xA_0^z \otimes B_0^xA_1^z$	$B_0^xA_1^z \otimes B_1^xA_0^z$	$B_0^xA_1^z \otimes B_0^xA_1^z$
ω_{yy}	*NA*	*QD*	*QD*	*QD*	$B_0^yA_0^z \otimes B_0^yA_0^z$	$B_0^yA_0^z \otimes B_0^yA_1^z$	$B_0^yA_1^z \otimes B_0^yA_0^z$	$B_0^yA_1^z \otimes B_0^yA_1^z$
$\omega_{y\bar{y}}$	*QD*	*NA*	*QD*	*QD*	$B_0^yA_0^z \otimes B_1^yA_0^z$	$B_0^yA_0^z \otimes B_1^yA_1^z$	$B_0^yA_1^z \otimes B_1^yA_0^z$	$B_0^yA_1^z \otimes B_1^yA_1^z$
$\omega_{\bar{y}y}$	*QD*	*QD*	*NA*	*QD*	$B_1^yA_0^z \otimes B_0^yA_0^z$	$B_1^yA_0^z \otimes B_0^yA_1^z$	$B_1^yA_1^z \otimes B_0^yA_0^z$	$B_1^yA_1^z \otimes B_0^yA_1^z$
$\omega_{\bar{y}\bar{y}}$	*QD*	*QD*	*QD*	*NA*	$B_1^yA_0^z \otimes B_1^yA_0^z$	$B_1^yA_0^z \otimes B_1^yA_1^z$	$B_1^yA_1^z \otimes B_1^yA_0^z$	$B_1^yA_1^z \otimes B_1^yA_1^z$

FIG. A.1 Each cell here denotes the explicit form of unitary $U_1 \otimes U_2$. For example $(B_0^xA_0^y \otimes B_0^xA_0^y)$ can be used to construct the measurement $\mathbf{M}' \equiv \{((B_0^xA_0^y)^\dagger \otimes (B_0^xA_0^y)^\dagger)E_1(B_0^xA_0^y \otimes B_0^xA_0^y), ((B_0^xA_0^y)^\dagger \otimes (B_0^xA_0^y)^\dagger)E_2(B_0^xA_0^y \otimes B_0^xA_0^y)\}$ to distinguish $|xx\rangle$ and $|yy\rangle$ perfectly.) *QD* stands for states which are distinguishable in Ordinary Quantum Composition. These states are definitely distinguishable in *SEP* composition since effects allowed in Quantum theory are also allowed in *SEP* theory. The cells named *NA* are invalid questions as $\eta' \neq \eta$ in $\mathbb{Q}(\eta, \eta')$.

Any pair of non-orthogonal states in $\Omega[12]$ can be perfectly distinguished with a measurement $\mathbf{M}' \equiv \{E_1', E_2'\}$ which is connected to the measurement $\mathbf{M}$ through some product unitary $U_i \otimes U_j$, where $U_i, U_j \in \mathbf{U}$. The measurement $\mathbf{M}'$ is therefore $\mathbf{M}' \equiv \Big\{(U_i^\dagger \otimes U_j^\dagger)E_1(U_i \otimes U_j),\ (U_i^\dagger \otimes U_j^\dagger)E_2(U_i \otimes U_j)\Big\}$. Proper choices of the unitaries for different pairs are listed in Figure [A.1]. $\square$

A.2 Proof of Theorem [7]

Proof. Let us denote the bipartite-elementary system in *SEP* as $\mathbb{C}^2 \otimes_{\min} \mathbb{C}^2$ whose state space is constructed through minimal tensor product of the state space of two qubit systems. We have already seen that there are at most twelve pairwise distinguishable states i.e., $\mathbf{I}_D(\mathbb{C}^2 \otimes_{\min} \mathbb{C}^2) = 12$. Let us denote these twelve states as $i \in \{1, \cdots, 12\}$, where any two of them such as $\{i, j\}$ can be pairwise distinguished through some elementary *SEP* measurement whenever $i \neq j$ & $i, j \in \{1, \cdots, 12\}$.

Consider k number of bipartite-elementary systems $(\mathbb{C}^2 \otimes_{\min} \mathbb{C}^2)^{\otimes_{\min} k}$ and the $2k$-parties product state

$$X = i_1 \otimes i_2 \otimes \cdots \otimes i_t \otimes \cdots \otimes i_k, \tag{A.5}$$

where $i_t \in \{1, \cdots, 12\}$ $\forall$ $t \in \{1, \cdots, k\}$ and t denotes the t^{th} bipartite-elementary SEP. Consider another such $2k$-parties product state

$$Y = j_1 \otimes j_2 \otimes \cdots \otimes j_t \otimes \cdots \otimes j_k, \tag{A.6}$$

where $j_t \in \{1, \cdots, 12\}$. The state X will be perfectly distinguishable to the state Y whenever $i_t \neq j_t$ for at least one $t \in \{1, \cdots, k\}$. For that particular t^{th} bipartite-elementary system Bob will follow the pairwise distinguishability strategy as discussed in the proof of Theorem 1. Therefore, for $2k$ number of *SEP*-bits we can construct a set of 12^k number of different states that are pairwise distinguishable by following the aforesaid procedure.

The construction above ensures that while playing the game $\mathscr{P}_D^{[12^k]}$ with elementary *SEP* bits, communication of $2k$ *SEP* bits from Alice to Bob suffices for a perfect winning strategy. Alice encodes her messages is the set of states constructed above and Bob decodes the messages based on the question given to him. It is not hard to see that $2k + \lceil k \log 3 \rceil$ number of qubits communication is necessary for winning this game. Here, $\lceil x \rceil$ denotes the ceiling function of x and $\log$ is in base 2. In other words, the advantage of *SEP* composition over quantum composition increases linearly with the increase in number of elementary systems. Here we would like to mention that the advantage we have reported might not be optimal. There is potential to get more advantage. Note that during the decoding step Bob addresses (at most) two elementary systems together in the strategy we have considered. There is a possibility that the number of pairwise distinguishable states will increase if Bob addresses all the systems together during his decoding process. □

Appendix B

Supplementary Material for Chapter [4]

B.1 Finding extremal entangled states in bipartite polygon models

Any unnormalized state in the maximal composition of the bipartite polygon system can be represented as a vector in $\mathbb{R}^9$, which can also be represented as a 3×3 matrix,

$$\Phi \equiv \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix}, \quad a_i \in \mathbb{R}. \tag{B.1}$$

Positivity of outcome probability demands $\mathrm{Tr}[e^{\mathrm{T}}(\Phi)] \geq 0$ for any effect allowed in this theory. Note that in maximal composition only product effects are allowed which forms an effect cone in $\mathbb{R}^9$, with the ray extremal effects $\mathscr{P}_{ef}[n]$ as defined in Eq.(4.1b). Therefore, the required positivity is assured once it is checked that $\mathrm{Tr}[e^T(\Phi)] \geq 0,\ \forall\ e \in \mathscr{P}_{ef}[n]$.

Normalization of the state is defined with the help of unit effect $u := u_A \otimes u_B^{\mathrm{T}}$, which demands $\mathrm{Tr}[u^{\mathrm{T}}(\Phi)] = 1$, and accordingly we have $a_9 = 1$. Thus a normalized state reads as

$$\Phi \equiv \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & 1 \end{pmatrix}, \tag{B.2}$$

and the set of normalized states forms a 8-dimensional polytope embedded in $\mathbb{R}^9$.

To find the extreme points of the normalized state space, first note that the inequality $\mathrm{Tr}[e(\Phi)] \geq 0$ represents an 8-dimensional half-space for any normalized state Φ and for any effect $e \in \mathscr{P}_{ef}[n]$. If we represent this expression as equality, *i.e.*, $\mathrm{Tr}[e(\Phi)] = 0$, then it corresponds to a 7-dimensional hyper-surface. Now for a different $e' \in \mathscr{P}_{ef}[n]$ the equation $\mathrm{Tr}[e'(\Phi)] = 0$ corresponds to another 7-dimensional hyper-surface which is either parallel to the hyper-surface corresponding to the effect e or they intersect each other in a 6-dimensional hyper-surface. Now for a third effect e'', if the 7-dimensional hyper-surface $\mathrm{Tr}[e''(\Phi)] = 0$ is not parallel to the hyper-surfaces corresponding to e and e' then it may intersect them in the same 6-dimensional hyper-surface (the intersection of $\mathrm{Tr}[e(\Phi)] = 0$ and $\mathrm{Tr}[e'(\Phi)] = 0$ hyper-surfaces) or these three intersect each other in a 5-dimensional hyper-surface (See Fig. [B.1] for visual representation). By proceeding this way, at some stage, 8 different hyper-surface corresponding to eight different effects from $\mathscr{P}_{ef}[n]$ will intersect at a single point, which corresponds to a normalized pure state. Mathematically this boils down to checking the uniqueness of the solution for a system of linear equations. The eight different effects from the set $\mathscr{P}_{ef}[n]$ can be chosen in ${}^{n^2}C_8$ different ways. All of these choices will not lead to a unique solution, but whenever it does, we obtain an extremal bipartite state for the maximal composition of the polygonal systems,

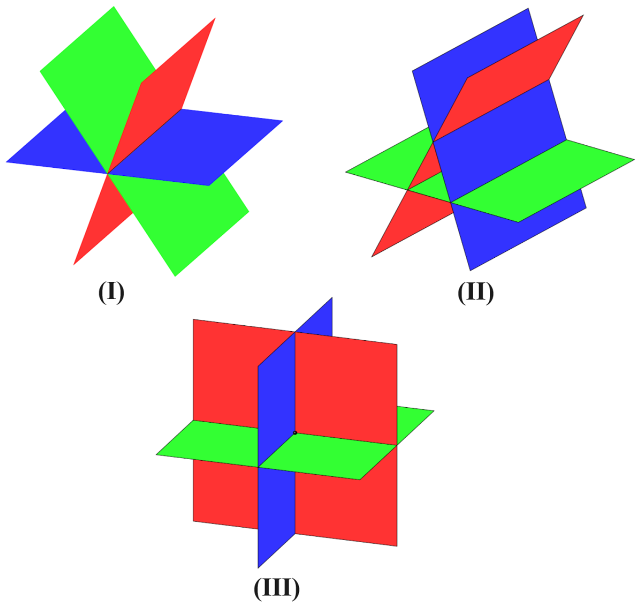

FIG. B.1 Three mutually non-parallel planes in $\mathbb{R}^3$ can intersect each other in three different ways. While in case (I) all three planes intersect in a common line, in case (II) each of the pairs intersect in different lines. On the other hand, in case (III) they intersect in a common unique point, which is of our interest.

provided the positivity constraints are satisfied. See the Appendix [B.1] for a more detailed discussion.

Once an extreme state Φ is identified, it is then straightforward to check whether it belongs to the set $\mathscr{P}_{st}[n]$ or not. If it does not belong to the set $\mathscr{P}_{st}[n]$, it corresponds to an extreme entangled state. Furthermore, the entangled states can be classified (see Definition 31) with the help of local reversible operations chosen from the set $\mathbf{T}_{AB}[n]$.

As discussed above, the amount of computations needed to find all the extreme states grows drastically with the number of extreme states of the elementary polygon systems. However, we can cut down this search space significantly by looking into the structure of the problem. Firstly we note that finding all extreme states is not necessary to find the entanglement classes. Once we have one state from each class, we can find all other extreme states by the action of the Local reversible transformations. Here we discuss a method to directly find a representative element for different entanglement classes.

As noted earlier atleast 8 hyperplanes are required to find out an extreme state. These hyperplane equations essentially represent positivity conditions. For the bipartite n-gon system let us denote the set of all extreme product effects as $\mathscr{P}_{ef}[n] \equiv \{E_1, \ldots, E_{n^2}\}$. Let the set $S_1 \equiv \{a_i\}_{i=1}^{8} \subset \mathscr{P}_{ef}[n]$ lead to a solution state (extreme) ω_1 and the set $S_2 \equiv \{b_j\}_{j=1}^{8} \subset \mathscr{P}_{ef}[n]$ lead to a solution state ω_2.

Definition 35. ***Local reversible equivalence:*** *Two sets S_1 and S_2 will be called equivalent under local reversible transformation (LRT) if $\exists\, s_1, s_2 \in \{+,-\}$ and $k_1, k_2 \in \{1, \ldots, n\}$ such that $S_2 = \bigcup_{i=1}^{8} \{(\mathbf{T}_{k_1}^{s_1} \otimes \mathbf{T}_{k_2}^{s_2}) a_i \mid a_i \in S_1\}$.*

In such a case the corresponding solutions ω_1 and ω_2 must also be *LR* equivalent, *i.e.*, $\omega_2 = (\mathbf{T}_{k_1}^{s_1} \otimes \mathbf{T}_{k_2}^{s_2})\omega_1$. To formally characterise this *LR* connected solutions in a systematic way we recall some preliminary concepts from group theory in the following subsection.

Preliminaries concepts from group theory

Let $\mathbf{G}$ be a finite group and let $O^{\mathbf{G}}$ denote a set of objects on which the group elements act. The set $O^{\mathbf{G}}$ is closed under action of group elements, *i.e.* $g(o_1) \in O^{\mathbf{G}}, \forall\, g \in \mathbf{G}$ & $o_1 \in O^{\mathbf{G}}$.

Definition 36 (Fixed point)**.** *An object $f \in O^{\mathbf{G}}$ is the fixed point of $g \in \mathbf{G}$ if it remains unchanged under the action of g, i.e. $g(f) = f$.*

Definition 37 (Orbit)**.** *The orbit* $\mathbf{O}_o$ *of an object* $o \in O^{\mathbf{G}}$ *is given by the set*

$$\mathbf{O}_o := \bigcup_{g \in \mathbf{G}} \{g(o)\}.$$

It is straightforward to see that if two objects o_1 and o_2 are related by some group action then $\mathbf{O}_{o_1} = \mathbf{O}_{o_2}$. Thus the set of all orbits partitions the collection of objects O into disjoint sets. Also, it can be noted that every object belongs to exactly one orbit. Here we recall the orbit-counting theorem by Burnside [162].

Lemma 4 (Burnside's Lemma)**.** *For a group* $\mathbf{G}$ *acting on a collection of objects* $O^{\mathbf{G}}$*, the number of orbits is given by*

$$\#\ \textit{of orbits} = \frac{1}{|\mathbf{G}|} \sum_{g \in \mathbf{G}} \#\ \textit{of fixed points of } g$$

where $|\mathbf{G}|$ *is the cardinality of the group* $\mathbf{G}$*.*

Group structure of LRT in bipartite polygon models

Since $\mathbf{T}_{k_1}^{s_1}$ represents rotations and reflections, *i.e.*, an element of dihedral group $\mathbf{D}_{2n}$, we can observe that $\left\{\mathbf{T}_{k_1}^{s_1} \otimes \mathbf{T}_{k_2}^{s_2} | s_1, s_2 \in \{+,-\} \text{ and } k_1, k_2 \in \{1,\ldots,n\}\right\}$ also forms a group which we denote by $\mathbf{D}_{2n}^{\times 2}$. We use this notation because this group is isomorphic to the group formed by the cartesian product of $\mathbf{D}_{2n}$ with itself. That is $\mathbf{D}_{2n}^{\times 2} \cong \mathbf{D}_{2n} \times \mathbf{D}_{2n}$. Now we define the set of objects $O^{\mathbf{D}_{2n}^{\times 2}}$ as

$$O^{\mathbf{D}_{2n}^{\times 2}} = \left\{\{a_1,\ldots,a_8\} \mid a_i \in \mathscr{P}_{ef}[n]\right\}.$$

An object $\{f_1,\ldots,f_8\}$ is called a fixed point of $\mathbf{T}_{k_1}^{s_1} \otimes \mathbf{T}_{k_2}^{s_2} \in \mathbf{D}_{2n}^{\times 2}$ if

$$\mathbf{T}_{k_1}^{s_1} \otimes \mathbf{T}_{k_2}^{s_2} (\{f_1,\ldots,f_8\}) = \{f_1,\ldots,f_8\}.$$

The orbit of an object $\{h_1,\ldots,h_8\} \in O^{\mathbf{D}_{2n}^{\times 2}}$ is given by

$$\mathbf{O}_{\{h_1,\ldots,h_8\}} = \bigcup_{g \in \mathbf{D}_{2n}^{\times 2}} \{g(\{h_1,\ldots,h_8\})\}.$$

Since objects of one orbit are not connected to objects of another orbit by LRT, therefore, instead of evaluating all the ${}^{n^2}C_8$ possible cases for finding valid solutions, we can restrict ourselves to evaluating just one object from each orbit. This helps us in reducing the search space drastically. A list comparing this

n	${}^{n^2}C_8$	# of orbits
4	12870	283
5	1081575	11103
6	30260340	213962

TABLE B.1 Number of orbits for bipartite polygon systems.

reduction is shown in Table [B.1]. Let $\mathbb{O}$ denote set of all orbits and let $x(\mathbf{O})$ be the representative object of an orbit $\mathbf{O} \in \mathbb{O}$. We can now define a set $\mathbf{X}_1$ as

$$\mathbf{X}_1 = \{x(\mathbf{O}) \mid \mathbf{O} \in \mathbb{O}\}.$$

Now we check whether the elements in $\mathbf{X}_1$ lead to a unique solution, *i.e.* whether the 8 hyperplanes corresponding to this object intersect at exactly one point. Using this we define the set $\mathbf{X}_2$ as

$$\mathbf{X}_2 = \{y_1 \in \mathbf{X}_1 \mid y_1 \text{ leads to a unique solution}\}.$$

The intersection of 8 planes doesn't necessarily imply that the solution satisfies all the positivity conditions since the intersection of the 8 hyper-planes could lie outside the state space. So we define $\mathbf{X}_3$ as

$$\mathbf{X}_3 = \{y_2 \in \mathbf{X}_2 \mid y_2 \text{ satisfies all positivity conditions}\}.$$

The elements in $\mathbf{X}_3$ are then analyzed to check if any of them are connected by local reversible transformations, which then lead to the end result of entangled states representative of each entanglement class.

Orbit counting: Box world

In the elementary Box world theory reversible transformations are the four rotations about the perpendicular axis passing through the centre, *i.e.* rotation about 0, $\pi/2$, π, and $3\pi/2$ radians; and four reflections (along the two diagonals and the two lines connecting the midpoints of the parallel sides). These four reflections can also be obtained by fixing only one reflections and then followed by four rotations. We denote the four different rotations by $\{\mathbb{I}, r, r^2, r^3\}$. Let us take the reflection that takes the effects e_0, e_1, e_2 and e_3 to e_3, e_2, e_1 and e_0, respectively to be f[1]. Then the four reflections are given by $\{f, rf, r^2 f, r^3 f\}$, yielding the full

[1]Please note that here the sub-index of e_i takes values from $\{0, \cdots, 3\}$, whereas in main manuscript it takes values from $\{1, \cdots, 4\}$. However, this does not change the orbit counting.

$\otimes$	e_0^B	e_1^B	e_2^B	e_3^B
e_0^A	1	2	3	4
e_1^A	5	6	7	8
e_2^A	9	10	11	12
e_3^A	13	14	15	16

TABLE B.2 The effect $e_i \otimes e_j$ is assigned a natural number following the rule $e_i \otimes e_j \to 4i+j+1$, where $i, j \in \{0,1,2,3\}$. For instance, $e_2^A \otimes e_3^B$ is assigned 12 (third row fourth column).

set of reversible transformations

$$\mathbf{R} = \left\{\mathbb{I}, r, r^2, r^3, f, rf, r^2 f, r^3 f\right\},$$

where *ab* operation implies operation *b* is followed by operation *a*. In the case of the composition of two such systems shared between Alice and Bob, we will denote the sets as

$$\mathbf{R_A} = \left\{\mathbb{I}_A, r_A, r_A^2, r_A^3, f_A, r_A f_A, r_A^2 f_A, r_A^3 f_A\right\},$$
$$\mathbf{R_B} = \left\{\mathbb{I}_B, r_B\ , r_B^2, r_B^3, f_B\ , r_B f_B\ , r_B^2 f_B\ , r_B^3 f_B\right\}.$$

Thus the set of all Local Reversible transformations on the composite system is given by

$$\mathbf{LR}_{box} = \{t_A \otimes t_B | t_A \in \mathbf{R_A},\ t_B \in \mathbf{R_B}\}.$$

Any product effect can be assigned a natural number using the rule $e_i \otimes e_j \to 4i+j+1$, where $i, j \in \{0,1,2,3\}$, which is shown in Table [B.2]. With this notation, if Alice applies the transformation r_A on her part then each row in Table [B.2] steps downward and the last row wraps back to the first row. Similarly, for r_B each column shifts one step rightwards and the last column wraps back to the first column. On the other hand, the operations f_A reflects the Table [B.2] about the central horizontal line, *i.e.* $\text{row}-1 \leftrightarrow \text{row}-4$ and $\text{row}-2 \leftrightarrow \text{row}-3$; and under f_B we have $\text{column}-1 \leftrightarrow \text{column}-4$ and $\text{column}-2 \leftrightarrow \text{column}-3$. All other local reversible transformations of Table [B.2] can be obtained by suitable combinations of these elementary operations (see Fig. [B.2]).

We now move on to calculating the number of orbits. For that, according to Burnside's Lemma, we need to find the number of fixed points for every local

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

(a)

13	14	15	16
1	2	3	4
5	6	7	8
9	10	11	12

(b)

4	1	2	3
8	5	6	7
12	9	10	11
16	13	14	15

(c)

13	14	15	16
9	10	11	12
5	6	7	8
1	2	3	4

(d)

4	3	2	1
8	7	6	5
12	11	10	9
16	15	14	13

(e)

5	8	7	6
1	4	3	2
13	16	15	14
9	12	11	10

(f)

FIG. B.2 Under the local reversible transformations ($\mathbf{LR}_{box}$), the arrangement of the numbers (associated with the product effect) in Table [B.2] gets modified. Here we show few examples: (a) $\mathbb{I}_A \otimes \mathbb{I}_B$, (b) $r_A \otimes \mathbb{I}_B$, (c) $\mathbb{I}_A \otimes r_B$, (d) $f_A \otimes \mathbb{I}_B$, (e) $\mathbb{I}_A \otimes f_B$, (f) $f_A r_A^2 \otimes f_B r_B^3$.

reversible transformation. Consider the transformation $r_A f_A \otimes \mathbb{I}_B$. As shown in Fig.B.3, the set of effects $\{1,2,3,4,9,10,11,12\}$ remain fixed under this transformation, whereas the pairs $\{5,13\}$,$\{6,14\}$,$\{7,15\}$ and $\{8,16\}$ exchange places among themselves. In order to find the number of fixed points of $r_A f_A \otimes \mathbb{I}_B$ we need to choose a set of 8 effects that remains invariant under the action of $r_A f_A \otimes \mathbb{I}_B$. We have the following possibilities:

1) choose all the fixed 8 effects $\implies {}^8C_8 \times {}^4C_0$ possibilities,

2) choose 6 fixed effects and 1 pair $\implies {}^8C_6 \times {}^4C_1$ possibilities,

3) choose 4 fixed effects and 2 pairs $\implies {}^8C_4 \times {}^4C_2$ possibilities,

4) choose 2 fixed effects and 3 pairs $\implies {}^8C_2 \times {}^4C_3$ possibilities,

5) choose all 4 pairs $\implies {}^8C_0 \times {}^4C_4$ possibilities,

Therefore we have a total of 646 different fixed points for $r_A f_A \otimes \mathbb{I}_B$. For the other group elements in $\mathbf{LR}_{box}$ we can carry a similar procedure to count the number of fixed points, which has been shown in Table [B.3]. Thus from Table [B.3] we

$\otimes$	$\mathbb{I}_B$	r_B	r_B^2	r_B^3	f_B	$r_B f_B$	$r_B^2 f_B$	$r_B^3 f_B$
$\mathbb{I}_A$	12870	6	70	6	70	646	70	646
r_A	6	6	6	6	6	6	6	6
r_A^2	70	6	70	6	70	70	70	70
r_A^3	6	6	6	6	6	6	6	6
f_A	70	6	70	6	70	70	70	70
$r_A f_A$	646	6	70	6	70	150	70	150
$r_A^2 f_A$	70	6	70	6	70	70	70	70
$r_A^3 f_A$	646	6	70	6	70	150	70	150

TABLE B.3 Number of fixed point for all the local reversible transformations $g = t_A \otimes t_B \in \mathbf{LR}_{box}$.

have the total number of orbits which turns out to be

$$\begin{aligned}
\#\text{ of orbits} &= \frac{1}{|\mathbf{LR}_{box}|} \sum_{g \in \mathbf{LR}_{box}} \#\text{ of fixed points of } g \\
&= \frac{1}{64} \times (\text{sum of all entries in Table [B.3]}) \\
&= \frac{18112}{64} = 283.
\end{aligned}$$

Following a similar counting procedure for higher gons, we obtain Table [B.1].

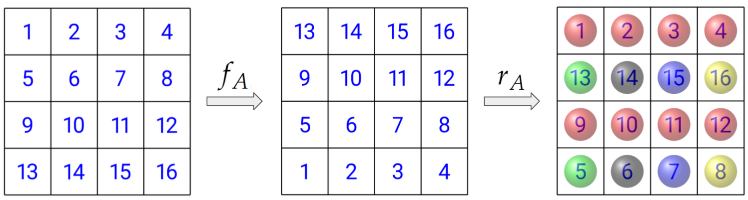

FIG. B.3 (Color online) The effects colored red remain fixed under the action of $r_A f_A \otimes \mathbb{I}_B$. The effects colored green $\{5, 13\}$ is a pair of effects which flip to each other upon the action of $r_A f_A \otimes \mathbb{I}_B$. Similarly, the black pair $\{6, 14\}$, the blue pair $\{7, 15\}$, and the yellow pair $\{8, 16\}$ flip to each other under action of $r_A f_A \otimes \mathbb{I}_B$.

B.2 More on Hardy Nonlocality of maximally entangled polygon states

B.2.1 Proof of Theorem [8]

Proof. The outcome probability for any pair of effect E on Alice's side and F on Bob's side for the maximally entangled state Φ_J of Eq.(4.2) read as

$$\begin{aligned} P_{\Phi_J}(E,F) &= \mathrm{Tr}\left[\left(E \otimes F^{\mathrm{T}}\right)\Phi_J^{\mathrm{T}}\right] \\ &= \mathrm{Tr}\left[E \otimes F^{\mathrm{T}}\right] = E \cdot F, \end{aligned} \tag{B.3}$$

where $E \cdot F$ be the usual inner product in $\mathbb{R}^3$. With this, the Hardy conditions of Eq.(2.53) become

$$P(E_1^{A+}, E_1^{B+} | \mathbf{M}_1^A\ \mathbf{N}_1^B) = E_1^{A+} \cdot E_1^{B+} > 0, \tag{B.4a}$$
$$P(E_1^{A+}, E_2^{B+} | \mathbf{M}_1^A\ \mathbf{N}_2^B) = E_1^{A+} \cdot E_2^{B+} = 0, \tag{B.4b}$$
$$P(E_2^{A+}, E_1^{B+} | \mathbf{M}_2^A\ \mathbf{N}_1^B) = E_2^{A+} \cdot E_1^{B+} = 0, \tag{B.4c}$$
$$P(E_2^{A-}, E_2^{B-} | \mathbf{M}_2^A\ \mathbf{N}_2^B) = E_2^{A-} \cdot E_2^{B-} = 0. \tag{B.4d}$$

To see whether the aforesaid conditions can be satisfied in any odd-gon, it is handy to have a look at the orthogonality graph for the extreme effects of the odd-gon theory. Two effects e and f will be called orthogonal to each other if and only if $e \cdot f = 0$. In any odd-gon theory, it turns out that a ray extremal effect e_r is orthogonal to only two other ray extremal effects $e_{s^\pm}$, with $s^\pm := \left(r + \frac{n\pm1}{2}\right) \bmod n$; and to the (non-ray) extremal effect $\bar{e}_r$. The orthogonality graph is shown in Fig. [B.4]. The proof of the theorem follows by analyzing the following two cases.

- **Case-I**: Let us assume that the effect E_2^{B+} in Eq.(B.4b) corresponds to some ray extremal effect (say) e_r. This implies $E_2^{B-} = \bar{e}_r$, since $\{E_2^{B+}, E_2^{B-}\}$ correspond to a measurement. Eq.(B.4d) further implies that E_2^{A-} must be orthogonal to E_2^{B-}. Since the only extreme effect orthogonal to $\bar{e}_r$ is e_r (see Fig. B.4), therefore we must have $E_2^{A-} = e_r$, which further implies $E_2^{A+} = \bar{e}_r$. Again Eq.(B.4c) implies $E_1^{B+} = e_r$ and hence $E_1^{A+} \cdot E_1^{B+} = E_1^{A+} \cdot E_2^{B+} = 0$, implying zero Hardy success.

- **Case-II**: Here we start by assuming that the effect E_2^{B+} in Eq.(B.4b) corresponds to some non ray extremal effect (say) $\bar{e}_r$. This implies $E_2^{B-} = e_r$, since $\{E_2^{B+}, E_2^{B-}\}$ correspond to a measurement. This also implies $E_1^{A+} = e_r$

from Eq.(B.4b). Now from Eq.(B.4d) we know that E_2^{A-} is orthogonal to E_2^{B-}. Which implies either $E_2^{A-} = e_{s^\pm}$ (with $s^\pm := \left(r + \frac{n\pm1}{2}\right) \bmod n$) or $E_2^{A-} = \bar{e}_r$. If $E_2^{A-} = e_{s^\pm}$ we have $E_2^{A+} = \bar{e}_{s^\pm}$ since $\{E_2^{A+}, E_2^{A-}\}$ forms a measurement. Which from Eq.(B.4c) further implies $E_1^{B+} = e_{s^\pm}$, yielding $E_1^{A+} \cdot E_1^{B+} = e_r \cdot e_{s^\pm} = 0$. Similarly if $E_2^{A-} = \bar{e}_r$ we would get $E_2^{A+} = e_r$ and $E_1^{A+} \cdot E_1^{B+} = E_2^{A+} \cdot E_1^{B+} = 0$. This proves that even for Case-II we have zero Hardy success probability, and hence this completes the proof.

□

B.2.2 Proof of Theorem [9]

Proof. For even-gons, all the extreme effects are ray extremal. The outcome probability $P_{\Phi_J}\left(e_i^A, e_j^B\right)$ of Alice's effect e_i^A and Bob's effect e_j^B on the maximally entangled state Φ_J of Eq.(4.3) reads as

$$\begin{aligned} P_{\Phi_J}\left(e_i^A, e_j^B\right) &= \mathrm{Tr}\left[\left(e_i \otimes e_j^{\mathrm{T}}\right)\Phi_J\right] \\ &= \frac{1}{4}\left[\sec\left(\frac{\pi}{n}\right)\cos\left(\frac{2(i-j)\pi}{n} - \frac{\pi}{n}\right)\right] + \frac{1}{4}. \end{aligned}$$

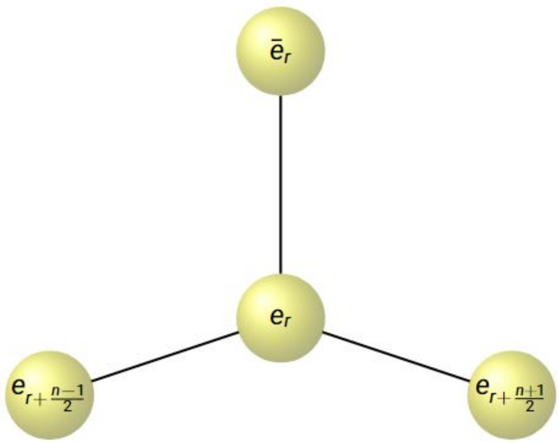

FIG. B.4 (Color online) Orthogonality graph of extreme effects for odd-gon theories. Each node denotes an extreme effect. Two effects e and f are connected with each other by an edge if and only if they are orthogonal to each other in the sense that $e \cdot f = 0$; here, the inner product is standard $\mathbb{R}^3$ inner product. While calculating the sub-indices of the effects modulo n operation is assumed throughout. Here, in order to be consistent with our notation we define 'r mod n' in such a way that it returns the remainder if the remainder is nonzero, otherwise it returns n.

Now $P_{\Phi_J}\left(e_i^A, e_j^B\right) = 0$ if and only if

$$\cos\left(\frac{2(i-j)\pi}{n} - \frac{\pi}{n}\right) = -\cos\left(\frac{\pi}{n}\right)$$
$$i.e.,\ i-j = \frac{1}{2} + \frac{n \pm 1}{2}, \tag{B.5}$$

where in the last expression modulo n addition is implied. Denoting $E_{1/2}^{A+/-}$ and $E_{1/2}^{B+/-}$ as the effects of Alice and Bob we can rewrite the Hardy nonlocal conditions (2.53) as

$$P(E_1^{A+}, E_1^{B+} |\ \mathbf{M}_1^A\ \mathbf{N}_1^B) > 0, \tag{B.6a}$$
$$P(E_1^{A+}, E_2^{B+} |\ \mathbf{M}_1^A\ \mathbf{N}_2^B) = 0, \tag{B.6b}$$
$$P(E_2^{A+}, E_1^{B+} |\ \mathbf{M}_2^A\ \mathbf{N}_1^B) = 0, \tag{B.6c}$$
$$P(E_2^{A-}, E_2^{B-} |\ \mathbf{M}_2^A\ \mathbf{N}_2^B) = 0. \tag{B.6d}$$

Let us consider E_1^{A+} in Eq.(B.6) be some ray extremal effect e_r for some $r \in \{1,2,\cdots,n\}$. Eq.(B.5) implies that to satisfy the condition of Eq.(B.6b), we must have $E_2^{B+} = e_{s(\alpha)}$, where $s(\alpha) = r - \frac{(n+1)}{2} + (-1)^\alpha \frac{1}{2}$, with Greek indices taking values from $\{0,1\}$. Since E_2^{B+} and E_2^{B-} forms a measurement, therefore we have $E_2^{B-} = e_{t(\alpha)}$, where $t(\alpha) = r - \frac{1}{2} + (-1)^\alpha \frac{1}{2}$. Again, Eqs.(B.5) and (B.6d) imply $E_2^{A-} = e_{v(\alpha,\beta)}$, where $v(\alpha,\beta) = r + [(-1)^\alpha + (-1)^\beta]\frac{1}{2} + \frac{n}{2}$, and $E_2^{A+} = e_{w(\alpha,\beta)}$, with $w(\alpha,\beta) = r + [(-1)^\alpha + (-1)^\beta]\frac{1}{2}$. Finally, Eqs.(B.5) and (B.6c) yield $E_1^{B+} := e_{z(\alpha,\beta,\gamma)} = r + [(-1)^\alpha + (-1)^\beta + (-1)^\gamma]\frac{1}{2} - \frac{1}{2} - \frac{n}{2}$. In other words, given the choice $E_1^{A+} = e_r$ the effect E_1^{B+} has only the following four choices

$$E_1^{B+} = e_{z(\alpha)};\ z(\alpha) := r - \frac{(n+1)}{2} + (-1)^\alpha \frac{1}{2}, \tag{B.7a}$$
$$E_1^{B+} = e_{z(\alpha)};\ z(\alpha) := r - \frac{(n+1)}{2} + (-1)^\alpha \frac{3}{2}. \tag{B.7b}$$

For the case of Eq.(B.7a), we have

$$
\begin{aligned}
&P_{\Phi_J}\left(E_1^{A+}, E_1^{B+}\right) \\
&= \frac{1}{4}\left[\sec\left(\frac{\pi}{n}\right)\cos\left(\frac{2(r-z(\alpha))\pi}{n} - \frac{\pi}{n}\right)\right] + \frac{1}{4} \\
&= \frac{1}{4}\left[\sec\left(\frac{\pi}{n}\right)\cos\left(\frac{2(\frac{n}{2}+\frac{1}{2}\pm\frac{1}{2})\pi}{n} - \frac{\pi}{n}\right)\right] + \frac{1}{4} \\
&= \frac{1}{4}\left[\sec\left(\frac{\pi}{n}\right)\cos\left(\pi \pm \frac{\pi}{n}\right)\right] + \frac{1}{4} \\
&= \frac{1}{4}\left[-\sec\left(\frac{\pi}{n}\right)\cos\left(\frac{\pi}{n}\right)\right] + \frac{1}{4} = 0.
\end{aligned}
$$

Therefore, these particular choices of E_1^{B+} do not exhibit Hardy nonlocality. However, for the choices of Eq.(B.7b) we obtain

$$
\begin{aligned}
&P_{\Phi_J}\left(E_1^{A+}, E_1^{B+}\right) \\
&= \frac{1}{4}\left[\sec\left(\frac{\pi}{n}\right)\cos\left(\frac{2(r-z(\alpha))\pi}{n} - \frac{\pi}{n}\right)\right] + \frac{1}{4} \\
&= \frac{1}{4}\left[\sec\left(\frac{\pi}{n}\right)\cos\left(\frac{2(\frac{n}{2}+\frac{1}{2}\pm\frac{3}{2})\pi}{n} - \frac{\pi}{n}\right)\right] + \frac{1}{4} \\
&= \frac{1}{4}\left[\sec\left(\frac{\pi}{n}\right)\cos\left(\pi \pm \frac{3\pi}{n}\right)\right] + \frac{1}{4} \\
&= \frac{1}{4}\left[-\sec\left(\frac{\pi}{n}\right)\cos\left(\frac{3\pi}{n}\right)\right] + \frac{1}{4} = \sin^2\left(\frac{\pi}{n}\right).
\end{aligned}
$$

This completes the proof of the theorem. □

B.3 More on Hardy Nonlocality of Mixed entangled polygon states

B.3.1 Proof of Lemma [2]

Proof. Consider a bipartite GPT in which the local parts have $OD = 2$ and satisfy preparation-measurement reciprocity. Let's assume there exists a pure product state $\omega_A \otimes \omega_B$ that can give rise to a Hardy-type local correlation when incompatible measurements $\{\mathbf{M}_1^A, \mathbf{M}_2^A\}$ and $\{\mathbf{N}_1^B, \mathbf{N}_2^B\}$ are performed on the subsystems. We denote these measurements as $\{e_i^k, u - e_i^k\}$ with $i \in \{1,2\}$ and $k \in \{A,B\}$. Without loss of any generality, we can assume that the effects $\{e_i^k, u - e_i^k\}$ are extreme. The preparation-measurement reciprocity demands that for a pure state ω on the local part there exists a unique extremal effect e_ω which filters the state ω perfectly. Now, if the theories with local OD$=2$ are considered, the extremal complementary effect corresponding to e_ω is given by $u-e_\omega$ which must be unique as well.[2]

Let us now, write down the equations for $\omega_A \otimes \omega_B$ to create a Hardy-type local correlation.

$$\begin{aligned} \mathrm{Tr}(e_1^A \omega_A) \times \mathrm{Tr}(e_1^B \omega_B) &= 0 \\ \mathrm{Tr}(e_1^A \omega_A) \times \mathrm{Tr}(e_2^B \omega_B) &= 0 \\ \mathrm{Tr}(e_2^A \omega_A) \times \mathrm{Tr}(e_1^B \omega_B) &= 0 \\ \mathrm{Tr}[(u-e_2^A)\omega_A] \times \mathrm{Tr}[(u-e_2^B)\omega_B] &= 0. \end{aligned}$$

Here the first equality follows from the fact that this correlation must be local. Let $\mathrm{Tr}(e_1^A \omega_A) = 0$, then preparation-measurement reciprocity, together with the fact that the OD of the local systems is exactly 2, implies $\mathrm{Tr}(e_2^A \omega_A), \mathrm{Tr}[(u-e_2^A)\omega_A] \neq 0$. Thus we must have $\mathrm{Tr}(e_1^B \omega_B) = \mathrm{Tr}[(u-e_2^B)\omega_B] = 0$. This is a contradiction as system B also obeys preparation-measurement reciprocity. A similar contradiction will arise if we assume $\mathrm{Tr}(e_1^B \omega_B) = 0$. This completes the proof. □

B.3.2 Proof of Theorem [10]

Proof. A local correlation cannot exhibit Hardy nonlocality by construction. However, there are deterministic local correlations that satisfy the zero conditions for Hardy nonlocality.

If our mixed state always consists of a pure product state for a preparation-measurement reciprocal theory, it follows from Lemma 2 that either $\mathbf{M}_1^A$ and $\mathbf{M}_2^A$ or $\mathbf{N}_1^B$ and $\mathbf{N}_2^B$ are the compatible pair of measurements.Under such a condition no entangled state can exhibit Hardy nonlocality, otherwise, it leads to superluminal communication [163–165]. This completes the proof. □

B.3.3 Proof of Theorem [11]

Proof. From Theorem [9] it follows that to obtain a Hardy nonlocal behaviour from the state Φ_J it requires $E_1^{A+} = e_r$ then $E_2^{A+} = e_{r\pm1}$. In an even-gon theory every two consecutive effects e_i and $e_{i\oplus_n 1}$ click with certainty on the state ω_i and

[2] One can interpret the preparation-measurement reciprocity theorem in an alternative way for the theories with OD$=2$: for every pure state ω, $(u-e_\omega)$ is the unique extreme effect that filters ω with zero probability. Note that, this does not hold good for OD≥ 3. For instance, one can consider bipartite qutrit systems which is not a trivial Hardy-local theory, in spite of admitting preparation-measurement reciprocity.

they never click on state $\omega_{i\oplus_n n}$. Therefore, $\forall r \in \{1,2,\cdots,n\}$ there is a state $\omega_r \in \Omega^A$ in Alice's side, such that $\mathrm{Tr}((E_1^{A+})^T \omega_r) = \mathrm{Tr}((E_2^{A+})^T \omega_r) = 0$. Similarly, there is a state $\omega_s \in \Omega^B$, such that $\mathrm{Tr}((E_2^{B-})^T \omega_s) = 0$. Evidently, the state $\omega_r \otimes \omega_s$ satisfies all the conditions Eq. (B.4b)-(B.4d) and equals to zero for Eq. (B.4a). So, for the state $W_\varepsilon = \varepsilon \Phi_J + (1-\varepsilon)\omega_r \otimes \omega_s$, with $\varepsilon \in (0,1]$, all the conditions (B.4b) - (B.4d) are satisfied, with $\mathrm{Tr}[(E_1^{A+} \otimes E_1^{B+})^{\mathrm{T}} W_\varepsilon] = \varepsilon \times \sin^2(\frac{\pi}{n})$. Clearly, for $\varepsilon \neq 1$, the state W_ε is a mixed state and hence establishes the claim of the theorem. □

B.3.4 Proof of Theorem [12]

Proof. If we consider two incompatible measurements $\mathbf{M}_1 \equiv \{e_1, \bar{e}_1\}$ and $\mathbf{M}_2 \equiv \{\bar{e}_2, e_2\}$ on Alice's part and two incompatible measurements $\mathbf{N}_1 \equiv \{e_1, \bar{e}_1\}$ and $\mathbf{N}_2 \equiv \{\bar{e}_2, e_2\}$ on Bob's part, then the resulting correlation obtained from the state W_ε depicts Hardy's nonlocality, whenever $\omega_i \otimes \omega_j \in \{\omega_3 \otimes \omega_4, \omega_4 \otimes \omega_3, \omega_4 \otimes \omega_4\}$. For $\omega_i \otimes \omega_j = \omega_3 \otimes \omega_5$, we require the measurements $\mathbf{M}_1 \equiv \{e_1, \bar{e}_1\}$, $\mathbf{M}_2 \equiv \{e_5, \bar{e}_5\}$ on Alice's part and $\mathbf{N}_1 \equiv \{e_1, \bar{e}_1\}$, $\mathbf{N}_2 \equiv \{\bar{e}_2, e_2\}$ on Bob's part; and for $\omega_i \otimes \omega_j = \omega_5 \otimes \omega_3$, $\mathbf{M}_1 \equiv \{e_1, \bar{e}_1\}$, $\mathbf{M}_2 \equiv \{\bar{e}_2, e_2\}$ on Alice's side and $\mathbf{N}_1 \equiv \{e_1, \bar{e}_1\}$, $\mathbf{N}_2 \equiv \{e_5, \bar{e}_5\}$ on Bob's side suffice the purpose. The success probability turns out to be $P^{succ} = \varepsilon \times (1 - \frac{4\sqrt{5}}{10}) \approx 0.1056\varepsilon$. □

Appendix C

Supplementary Material for Chapter [5]

C.1 Proof of Theorem [14]

Proof. The proof is divided into two parts:

(i) *only if* part: $P_{succ}^{\text{DR-B}} = \mu$ ensures a protocol for GYNI game yielding success probability $P_{succ}^{\text{GYNI}} = \mu$.

(ii) *if* part: $P_{succ}^{\text{GYNI}} = \mu$ ensures a protocol for DR-B task yielding success probability $P_{succ}^{\text{DR-B}} = \mu$.

only if part:
Given the encoded states $\{\mathscr{B}_{AB}^{\mathbf{x}}\}$, let the process matrix $W_{A_IA_OB_IB_O}$ yields a success $P_{succ}^{\text{DR-B}} = \mu$ with Alice and Bob applying the quantum instruments $\mathbf{I_A} = \{M_{AA_IA_O}^a\}_{a=0}^1$ and $\mathbf{I_B} = \{M_{BB_IB_O}^b\}_{b=0}^1$, respectively. Thus we have,

$$P_{succ}^{\text{DR-B}} = \frac{1}{4}\sum_{x_1,x_2=0}^{1} p(a=x_1, b=x_2|\mathscr{B}_{AB}^{x_1x_2}) = \mu, \quad \text{with,} \tag{C.1}$$
$$p(a,b|\mathscr{B}_{AB}^{x_1x_2}) := \text{Tr}[(\mathscr{B}_{AB}^{x_1x_2} \otimes W)(M_{AA_IA_O}^a \otimes M_{BB_IB_O}^b)].$$

For playing the *GYNI* game, let Alice and Bob share the process Matrix $W' = W_{A_IA_OB_IB_O} \otimes \mathscr{B}_{AB}^{00}$. Based on their coin states $i_1, i_2 \in \{0,1\}$, Alice and Bob respectively perform quantum instruments

$$\mathbf{I_A}^{(i_1)} := \left\{\mathbb{Z}_A^{i_1}\left(M_{AA_IA_O}^a\right)\right\}_{a=0}^1 \equiv \left\{Z_A^{i_1} M_{AA_IA_O}^a Z_A^{i_1}\right\}_{a=0}^1,$$
$$\mathbf{I_B}^{(i_2)} := \left\{\mathbb{X}_B^{i_2}\left(M_{BB_IB_O}^{b|i_2}\right)\right\}_{b=0}^1 \equiv \left\{X_B^{i_2} M_{BB_IB_O}^b X_B^{i_2}\right\}_{b=0}^1,$$

where Z & X are qubit Pauli gates and $\{M_{AA_IA_O}^a\}_{a=0}^1$ & $\{M_{BB_IB_O}^b\}_{b=0}^1$ are the instruments used in DR-B task. The success probability of GYNI game, therefore, reads as

$$\begin{aligned}
P_{succ}^{\text{GYNI}} &= \sum_{i_1,i_2=0}^{1} \frac{1}{4} p(a=i_2, b=i_1 | i_1, i_2) \\
&= \frac{1}{4} \sum_{i_1,i_2=0}^{1} \operatorname{Tr}\left[\left(\mathscr{B}_{AB}^{00} \otimes W\right) \left(M_{AA_IA_O}^{a=i_2|i_1} \otimes M_{BB_IB_O}^{b=i_1|i_2} \right) \right] \\
&= \frac{1}{4} \sum_{i_1,i_2=0}^{1} \operatorname{Tr}\left[\left(\mathscr{B}_{AB}^{00} \otimes W\right) \left(\mathbb{Z}_A^{i_1} \left(M_{AA_IA_O}^{a=i_2} \right) \otimes \mathbb{X}_B^{i_2} \left(M_{BB_IB_O}^{b=i_1} \right) \right) \right] \\
&= \frac{1}{4} \sum_{i_1,i_2=0}^{1} \operatorname{Tr}\left[\left(\left(\mathbb{Z}^{i_1} \otimes \mathbb{X}^{i_2} \left(\mathscr{B}^{00} \right) \right) \otimes W \right) \left(M_{AA_IA_O}^{a=i_2} \otimes M_{BB_IB_O}^{b=i_1} \right) \right] \\
&= \frac{1}{4} \sum_{i_1,i_2=0}^{1} \operatorname{Tr}\left[\left(\mathscr{B}_{AB}^{i_2 i_1} \otimes W \right) \left(M_{AA_IA_O}^{a=i_2} \otimes M_{BB_IB_O}^{b=i_1} \right) \right] \\
&= \mu = P_{succ}^{\text{DR-B}}, \quad [\text{using Eq.}(C.1)].
\end{aligned} \tag{C.2}$$

This completes the *only if* part of the claim.

if part:
Given x_1 and x_2 being the respective coin states of Alice and Bob, let the process matrix $W'_{A_IA_OB_IB_O}$ yields a success $P_{succ}^{\text{GYNI}} = \mu$, with Alice and Bob performing quantum instruments $\mathbf{I_A}^{(x_1)} = \{M_{A_IA_O}^{a|x_1}\}_{a=0}^{1}$ and $\mathbf{I_B}^{(x_2)} = \{M_{B_IB_O}^{b|x_2}\}_{b=0}^{1}$, respectively. Thus we have,

$$\begin{aligned}
P_{succ}^{\text{GYNI}} &= \frac{1}{4} \sum_{x_1,x_2=0}^{1} p(a=x_2, b=x_1 | x_1, x_2) = \mu, \quad \text{with,} \\
p(a,b|x_1,x_2) &:= \operatorname{Tr}\left[\left(M_{A_IA_O}^{a|x_1} \otimes M_{B_IB_O}^{b|x_2} \right) W' \right].
\end{aligned} \tag{C.3}$$

To perform the DR-B task, Alice and Bob share the Process Matrix $W'_{A_IA_OB_IB_O} \otimes \mathscr{B}_{A'B'}^{00}$. Now, given the encoded state $\mathscr{B}_{AB}^{x_1x_2}$, Alice and Bob apply the following unitary operation on parts of their local systems

$$U_{AA'} = U_{BB'} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix}, \tag{C.4}$$

which results in

$$\begin{aligned}
&\mathscr{U}_{AA'} \otimes \mathscr{U}_{BB'} \left(W'_{A_IA_OB_IB_O} \otimes \mathscr{B}_{A'B'}^{00} \otimes \mathscr{B}_{AB}^{x_1x_2} \right) \\
&\quad = W'_{A_IA_OB_IB_O} \otimes \mathscr{B}_{A'B'}^{x_10} \otimes \mathscr{B}_{AB}^{x_2x_1},
\end{aligned} \tag{C.5}$$

where $\mathscr{U}(\rho) := U\rho U^\dagger$. On A and A' sub-parts of the evolved process Alice performs computational basis measurement (*i.e.* the Pauli-σ^3 measurement), resulting into outcomes $u, u' \in \{0,1\}$, where 0 (1) corresponds to 'up' ('down') outcome. Bob also does the same resulting into outcomes $v, v' \in \{0,1\}$. Clearly, due to correlation of the state, we have

$$u \oplus v = x_2, \;\; \& \;\; u' \oplus v' = x_1. \tag{C.6}$$

Therefore, guessing the value of v' and u respectively by Alice and Bob with the probability μ will ensure the same success in DR-B task. At this point the process $W'_{A_I A_O B_I B_O}$ proves to be helpful in this task, which can be accordingly chosen looking into its advantage in GYNI game. Rest of the protocol mimics GYNI strategy with u and v' being the inputs of Alice and Bob, respectively. Denoting a' and b' as the output of the GYNI strategy the final guess in DR-B task by Alice and Bob are respectively,

$$a = a' \oplus u', \;\; \& \;\; b = b' \oplus v. \tag{C.7}$$

On the composite process $W'_{A_I A_O B_I B_O} \otimes \mathscr{B}^{00}_{A'B'} \otimes \mathscr{B}^{x_1 x_2}_{AB}$, the effective instruments $\{M^a_{AA'A_I A_O}\}^1_{a=0}$ for Alice and $\{M^b_{BB'B_I B_O}\}^1_{b=0}$ for Bob are respectively given by

$$\sum_{u,u',a'=0}^{1} \delta_{a,a'\oplus u'} \mathscr{U}_{AA'} \otimes \mathbf{id}_{A_I A_O} (|uu'\rangle_{AA'} \langle uu'| \otimes M^{a'|u}_{A_I A_O}),$$
$$\sum_{v,v',b'=0}^{1} \delta_{b,b'\oplus v} \mathscr{U}_{BB'} \otimes \mathbf{id}_{B_I B_O} (|vv'\rangle_{BB'} \langle vv'| \otimes M^{b'|v'}_{B_I B_O}),$$

where, $\{M^{a'|u}_{A_I A_O}\}$ and $\{M^{b'|v'}_{B_I B_O}\}$ are the instruments used by Alice and Bob in GYNI game. The success probability of DR-B task with the aforesaid protocol turns out to be

$$\begin{aligned}
P_{succ}^{\text{DR-B}} &= \sum_{x_1,x_2=0}^{1} \frac{1}{4} P(a=x_1, b=x_2|\mathscr{B}_{AB}^{x_1x_2}) \\
&= \sum_{x_1,x_2=0}^{1} \frac{1}{4} \operatorname{Tr}\left[\left(W' \otimes \mathscr{B}^{00} \otimes \mathscr{B}^{x_1x_2}\right)\left(M_{AA'A_IA_O}^{a=x_1} \otimes M_{BB'B_IB_O}^{b=x_2}\right)\right] \\
&= \sum_{\substack{x_1,x_2,u,u',\\ a',v,v',b'=0}}^{1} \frac{1}{4} \delta_{a=x_1,a'\oplus u'} \delta_{b=x_2,b'\oplus v} \operatorname{Tr}\left[\left(W' \otimes \mathscr{B}_{A'B'}^{x_10} \otimes \mathscr{B}_{AB}^{x_2x_1}\right)\right. \\
&\qquad \left.\left(|uvu'v'\rangle_{ABA'B'} \langle uvu'v'| \otimes M_{A_IA_O}^{a'|u} \otimes M_{B_IB_O}^{b'|v'}\right)\right] \\
&= \sum_{\substack{x_1,x_2,u,u',\\ a',v,v',b'=0}}^{1} \frac{1}{16} \delta_{x_1,a'\oplus u'} \delta_{x_2,b'\oplus v} \delta_{x_2,u\oplus v} \delta_{x_1,u'\oplus v'} \\
&\qquad \operatorname{Tr}\left[(W' \left(M_{A_IA_O}^{a'|u} \otimes M_{B_IB_O}^{b'|v'}\right)\right] \\
&= \sum_{\substack{x_1,x_2,u,u',\\ a',v,v',b'=0}}^{1} \frac{1}{16} \delta_{x_1,a'\oplus u'} \delta_{x_2,b'\oplus v} \delta_{x_2,u\oplus v} \delta_{x_1,u'\oplus v'} P(a',b'|u,v') \\
&= \sum_{\substack{x_1,x_2,u,u',\\ v,v'=0}}^{1} \frac{1}{16} \delta_{x_2,u\oplus v} \delta_{x_1,u'\oplus v'} P(x_1 \oplus u', x_2 \oplus v|u,v') \\
&= \sum_{x_1,x_2,u,v'=0}^{1} \frac{1}{16} P(x_1 \oplus v' \oplus x_1, x_2 \oplus u \oplus x_2|u,v') \\
&= \sum_{x_1,x_2} \frac{1}{4} \sum_{u,v'} \frac{1}{4} P(v',u|u,v') = \sum_{x_1,x_2} \frac{1}{4} \mu = \mu \;\; [\text{using eq.}(C.3)]\,.
\end{aligned} \tag{C.8}$$

This completes the *if* part of the claim, and hence the Theorem is proved. □

C.2 Locally Inaccessible Data Retrieval from Maximally Entangled States

In 5, we observed a strict duality between the success probability of the bipartite DR-B task and the success probability of the GYNI game. In this section, we will extend this concept of Theorem [14] by Hiding Dits in higher dimensional Maximally Entangled States(DR-ME) and GYNI with dit inputs.

DR-ME: Referee encodes the string $\mathbf{x} = x_1\ x_2 \in \{0,1,\cdots,d-1\}^2$ in bipartite maximally entangles states as follows:

$$\mathbf{x} \to |\mathscr{B}^{\mathbf{x}}\rangle_{AB} := \frac{1}{\sqrt{d}}\sum_{k=0}^{d-1}\omega^{x_2 k}|k\rangle_A \otimes |k \oplus_d x_1\rangle_B,$$

$$= \left(Z_A^{x_2} \otimes X_B^{x_1}\right)\frac{1}{\sqrt{d}}\sum_{k=0}^{d-1}|k\rangle_A \otimes |k\rangle_B, \tag{C.9a}$$

$$Z_A|k\rangle := e^{\frac{2\pi i k}{d}}|k\rangle, \ \ \& \ \ X_A|k\rangle := |k \oplus_d 1\rangle \tag{C.9b}$$

with $\omega = e^{\frac{2\pi i}{d}}$ and $\oplus_d$ representing modulo d addition. The local marginals of Alice and Bob are the maximally mixed states $\mathbf{I}/d$ for every encoded state, and thus the hiding condition is satisfied. The success probability of DR-ME task is given by

$$P_{succ}^{\text{DR-ME}} := \sum_{x_1,x_2=0}^{d-1}\frac{1}{d^2}P(a = x_1, b = x_2|\mathscr{B}_{AB}^{x_1x_2}). \tag{C.10}$$

GYNI-d: Alice (Bob) tosses a random d sided coin to generate a random dit i_1 $(i_2) \in \{0,1,\cdots,d-1\}$. Each party aims to guess the coin state of the other party. Denoting their guesses as a and b respectively, the success probability reads as

$$P_{succ}^{\text{GYNI-d}} = \sum_{i_1,i_2=0}^{d-1}\frac{1}{d^2}P(a = i_2, b = i_1|i_1,i_2). \tag{C.11}$$

The optimal winning probabilities for GYNI-d with an indefinite causal ordered process are unknown. However, the duality established in Theorem **??** extends to this higher dimensional case.

Theorem 23. *A success probability $P_{succ}^{DR\text{-}ME} = \mu$ in DR-ME task is achievable if and only if the same success is achievable in GYNI-d game, i.e., $P_{succ}^{GYNI\text{-}d} = \mu$.*

Proof. As before the proof is done in two parts:

(i) *only if* part: $P_{succ}^{\text{DR-ME}} = \mu$ ensures a protocol for GYNI-d game yielding success probability $P_{succ}^{\text{GYNI-d}} = \mu$.

(ii) *if* part: $P_{succ}^{\text{GYNI-d}} = \mu$ ensures a protocol for DR-ME task yielding success probability $P_{succ}^{\text{DR-ME}} = \mu$.

only if part:
Given the encoded states $\{\mathscr{B}_{AB}^{\mathbf{x}}\}$, let the process matrix $W_{A_IA_OB_IB_O}$ yields a suc-

cess $P_{succ}^{\text{DR-ME}} = \mu$ with Alice and Bob applying the quantum instruments $\mathbf{I_A} = \{M^a_{AA_IA_O}\}_{a=0}^{d-1}$ and $\mathbf{I_B} = \{M^b_{BB_IB_O}\}_{b=0}^{d-1}$, respectively. Thus we have,

$$P_{succ}^{\text{DR-ME}} = \frac{1}{d^2}\sum_{x_1,x_2=0}^{d-1} p(a=x_1, b=x_2|\mathscr{B}_{AB}^{x_1x_2}) = \mu, \text{ with,} \quad \text{(C.12)}$$
$$p(a,b|\mathscr{B}_{AB}^{x_1x_2}) := \text{Tr}[(\mathscr{B}_{AB}^{x_1x_2} \otimes W)(M^a_{AA_IA_O} \otimes M^b_{BB_IB_O})].$$

For playing the GYNI-d game, let Alice and Bob share the process Matrix $W' = W_{A_IA_OB_IB_O} \otimes \mathscr{B}^{00}_{AB}$. Based on their coin states $i_1, i_2 \in \{0,1,\cdots,d-1\}$, Alice and Bob respectively perform quantum instruments

$$\mathbf{I_A}^{(i_1)} := \left\{\mathbb{Z}_A^{i_1}\left(M^a_{AA_IA_O}\right)\right\}_{a=0}^{d-1} \equiv \left\{Z_A^{i_1} M^a_{AA_IA_O} Z_A^{i_1}\right\}_{a=0}^{d-1},$$
$$\mathbf{I_B}^{(i_2)} = \left\{\mathbb{X}_B^{i_2}\left(M^b_{BB_IB_O}\right)\right\}_{b=0}^{d-1} \equiv \left\{X_B^{i_2} M^b_{BB_IB_O} X_B^{i_2}\right\}_{b=0}^{d-1},$$

where Z & X are Pauli gates on $\mathbb{C}^d$ as defined in eq.(C.9b) and $\{M^a_{AA_IA_O}\}_{a=0}^{d-1}$ & $\{M^b_{BB_IB_O}\}_{b=0}^{d-1}$ are the instruments used in DR-ME task. With this protocol the success probability of GYNI-d game becomes

$$\begin{aligned}
P_{succ}^{\text{GYNI-d}} &= \sum_{i_1,i_2=0}^{d-1} \frac{1}{d^2} p(a=i_2, b=i_1|i_1,i_2) \\
&= \frac{1}{d^2}\sum_{i_1,i_2=0}^{d-1} \text{Tr}\left[\left(\mathscr{B}^{00}_{AB} \otimes W\right)\left(M^{a=i_2|i_1}_{AA_IA_O} \otimes M^{b=i_1|i_2}_{BB_IB_O}\right)\right] \\
&= \frac{1}{d^2}\sum_{i_1,i_2=0}^{d-1} \text{Tr}\left[\left(\mathscr{B}^{00}_{AB} \otimes W\right)\left(\mathbb{Z}_A^{i_1}\left(M^{a=i_2}_{AA_IA_O}\right) \otimes \mathbb{X}_B^{i_2}\left(M^{b=i_1}_{BB_IB_O}\right)\right)\right] \\
&= \frac{1}{d^2}\sum_{i_1,i_2=0}^{d-1} \text{Tr}\left[\left(\left(\mathbb{Z}^{i_1} \otimes \mathbb{X}^{i_2}\left(\mathscr{B}^{00}\right)\right) \otimes W\right)\left(M^{a=i_2}_{AA_IA_O} \otimes M^{b=i_1}_{BB_IB_O}\right)\right] \\
&= \frac{1}{d^2}\sum_{i_1,i_2=0}^{d-1} \text{Tr}\left[\left(\mathscr{B}^{i_2i_1}_{AB} \otimes W\right)\left(M^{a=i_2}_{AA_IA_O} \otimes M^{b=i_1}_{BB_IB_O}\right)\right] \\
&= \mu = P_{succ}^{\text{DR-B}}, \text{ [using Eq.}(C.1)].
\end{aligned}$$

This completes the *only if* part of the claim.

if part:

Given x_1 and x_2 being the respective coin states of Alice and Bob, let the process matrix $W'_{A_IA_OB_IB_O}$ yields a success $P_{succ}^{\text{GYNI}} = \mu$, with Alice and Bob performing quantum instruments $\mathbf{I_A}^{(x_1)} = \{M^{a|x_1}_{A_IA_O}\}_{a=0}^{d-1}$ and $\mathbf{I_B}^{(x_2)} = \{M^{b|x_2}_{B_IB_O}\}_{b=0}^{d-1}$, respectively. Thus we have,

$$P_{succ}^{\text{GYNI}} = \frac{1}{d^2}\sum_{x_1,x_2=0}^{d-1} p(a=x_2, b=x_1|x_1,x_2) = \mu, \quad \text{with,}$$
$$p(a,b|x_1,x_2) := \text{Tr}\left[\left(M_{A_IA_O}^{a|x_1} \otimes M_{B_IB_O}^{b|x_2}\right) W'\right]. \tag{C.13}$$

To perform the DR-ME task, Alice and Bob share the Process Matrix $W'_{A_IA_OB_IB_O} \otimes \mathscr{B}^{00}_{A'B'}$. Now, given the encoded state $\mathscr{B}^{x_1x_2}_{AB}$, Alice and Bob apply the following Controlled-Shift(CS) unitary operation on parts of their local systems

$$CS_{AA'}|m\rangle_A|n\rangle_{A'} = |m\rangle_A|n\oplus_d m\rangle_{A'} \tag{C.14}$$

They follow this with a discrete Fourier transformation, $F|k\rangle = \frac{1}{\sqrt{d}}\sum_{q=0}^{d-1}\omega^{qk}|q\rangle$, on their respective unprimed parts, which results in

$$\mathbb{F}_{A'}\circ\mathbb{CS}_{AA'}\otimes\mathbb{F}_{B'}\circ\mathbb{CS}_{BB'}\left(W'_{A_IA_OB_IB_O}\otimes\mathscr{B}^{00}_{A'B'}\otimes\mathscr{B}^{x_1x_2}_{AB}\right)$$
$$= W'_{A_IA_OB_IB_O}\otimes\mathscr{B}^{x_10}_{A'B'}\otimes\mathscr{B}^{x_1x_2}_{AB}, \tag{C.15}$$

where $\mathbb{F}$ and $\mathbb{CS}$ denote the linear maps corresponding to the unitary operations F and CS respectively. After this Alice performs computational basis measurement on A and A', resulting in outcomes $u,u' \in \{0,1,\cdots,d-1\}$. Similarly, Bob obtains the outcomes $v,v' \in \{0,1,\cdots,d-1\}$. Clearly, due to the correlation of the state, we have

$$u\oplus_d v = x_2, \quad \& \quad u'\oplus_d v' = x_1. \tag{C.16}$$

Therefore, guessing the value of v' and u respectively by Alice and Bob with the probability μ will ensure the same success in the DR-ME task. At this point the process $W'_{A_IA_OB_IB_O}$ proves to be helpful in this task, which can be accordingly chosen looking into its advantage in GYNI-d game. The rest of the protocol mimics the GYNI-d strategy with u and v' being the inputs of Alice and Bob, respectively. Denoting a' and b' as the output of the GYNI strategy the final guess in the DR-ME task by Alice and Bob are respectively,

$$a = a'\oplus_d u', \quad \& \quad b = b'\oplus_d v. \tag{C.17}$$

On the composite process $W'_{A_IA_OB_IB_O} \otimes \mathscr{B}^{00}_{A'B'} \otimes \mathscr{B}^{x_1x_2}_{AB}$, the effective instruments $\{M^a_{AA'A_IA_O}\}_{a=0}^{d-1}$ and $\{M^b_{BB'B_IB_O}\}_{b=0}^{d-1}$ are respectively given by

$$\sum_{u,u',a'=0}^{d-1} \delta_{a,a'\oplus_d u'} \mathbb{F}_{A'} \circ \mathbb{CS}_{AA'} \otimes \mathbf{id}_{A_IA_O}(|uu'\rangle_{AA'} \langle uu'| \otimes M^{a'|u}_{A_IA_O}),$$
$$\sum_{v,v',b'=0}^{d-1} \delta_{b,b'\oplus_d v} \mathbb{F}_{B'} \circ \mathbb{CS}_{BB'} \otimes \mathbf{id}_{B_IB_O}(|vv'\rangle_{BB'} \langle vv'| \otimes M^{b'|v'}_{B_IB_O}),$$

where $\{M^{a'|u}_{A_IA_O}\}$ and $\{M^{b'|v'}_{B_IB_O}\}$ are the instruments used by Alice and Bob in GYNI-d game. The success probability of the DR-ME task with the aforesaid protocol becomes

$$\begin{aligned} P^{\text{DR-B}}_{succ} &= \sum_{x_1,x_2=0}^{1} \frac{1}{d^2} P(a = x_1, b = x_2 | \mathscr{B}^{x_1x_2}_{AB}) \\ &= \sum_{x_1,x_2=0}^{d-1} \frac{1}{d^2} \operatorname{Tr}\left[\left(W' \otimes \mathscr{B}^{00} \otimes \mathscr{B}^{x_1x_2}\right) \left(M^{a=x_1}_{AA'A_IA_O} \otimes M^{b=x_2}_{BB'B_IB_O}\right)\right] \\ &= \sum_{x_1,x_2} \frac{1}{d^2} \sum_{u,v'} \frac{1}{d^2} P(v', u | u, v') \\ &= \sum_{x_1,x_2} \frac{1}{d^2} \mu = \mu. \ \text{[using Eq.}(C.13)]. \end{aligned} \tag{C.18}$$

This completes the *if* part of the claim, and hence the Theorem is proved. □

C.3 Advantage in T-DR from Indefinite Ordered Classical Processes

C.3.1 Causal Indefiniteness in Classical Setup

The state cone (Ω^n_+) and normalised state space (Ω^n) of an n level classical system is described as

$$\Omega^n_+ := \{\vec{p} \mid \vec{p} \in \mathbb{R}^n , p_i \geq 0 \ \forall \ i\}, \tag{C.19a}$$
$$\Omega^n := \left\{\vec{p} \mid \vec{p} \in \mathbb{R}^n , p_i \geq 0 \ \forall \ i , \& \sum_{i=0}^{n-1} p_i = 1\right\}. \tag{C.19b}$$

Pure state of Ω^n are $\vec{l} := \{\delta_{il}\}_{i=0}^{n-1}$ for $l \in \{0, \cdots, n-1\}$. Later, sometime we will denote $\vec{l} \equiv l$ simply. The most general operation that an agent (say X) can apply on a classical system is described by a classical instrument

$$\mathscr{I}^c_X \equiv \left\{S^k_X | S^k_X : \Omega^{I_X}_+ \mapsto \Omega^{O_X}_+\right\}_{k=1}^{N}, \tag{C.20}$$

where S_X^k are positive linear maps mapping the state cone of the input I_X level classical system to the state cone of the output O_X level classical system, with $I_X, O_X < \infty$. Moreover, S_X^k's sum up to a stochastic map $\mathbb{S}_X$, *i.e.*,

$$\mathbb{S}_X := \sum_{k=1}^{N} S_X^k, \; s.t. \; \mathbb{S}_X(\Omega^{I_X}) \subseteq \Omega^{O_X}. \tag{C.21}$$

The stochasticity condition is analogous to the trace preserving condition in the quantum case. Let us consider the case involving two parties say Alice and Bob with

$$\mathscr{S}_A := \{S_A \mid S_A : \Omega_+^{I_A} \mapsto \Omega_+^{O_A}\}, \tag{C.22a}$$

$$\mathscr{S}_B := \{S_B \mid S_B : \Omega_+^{I_B} \mapsto \Omega_+^{O_B}\}, \tag{C.22b}$$

denoting the sets of all state-cone preserving maps for Alice and Bob, respectively. Any such linear map $S : \Omega_+^n \mapsto \Omega_+^m$ can be represented as an $m \times n$ real matrix, which can be uniquely specified by it's action on pure states $\{l\}_{l=0}^{n-1}$ of Ω^n. Without assuming any background causal structure among Alice's and Bob's actions, the most general statistics observed is given by a bi-linear functional,

$$P : \mathscr{S}_A \times \mathscr{S}_B \mapsto [0, \infty), \tag{C.23a}$$

$$P(\mathbb{S}_A, \mathbb{S}_B) = 1, \; \forall \; \mathbb{S}_A, \mathbb{S}_B. \tag{C.23b}$$

Any such bi-linear functional reads as a Trace-rule over a stochastic map $\mathbb{E}_{AB}$ [114], *i.e.*,

$$P(S_A, S_B) = \mathrm{Tr}\left[\mathbb{E}_{AB}\left(S_A \otimes S_B\right)\right], \tag{C.24a}$$

$$\mathrm{Tr}[\mathbb{E}_{AB}(\mathbb{S}_A \otimes \mathbb{S}_B)] = 1, \; \forall \; \mathbb{S}_A, \mathbb{S}_B \tag{C.24b}$$

$$\mathbb{E}_{AB}(\Omega^{O_A O_B}) \subseteq \Omega^{I_A I_B}. \tag{C.24c}$$

Such a $\mathbb{E}_{AB}$ is termed as logically-consistent-classical process (LCCP). As shown in [114] (see also [11]) all bipartite LCCPs are causally definite, *i.e.*,

$$\mathbb{E}_{AB} = p_1 \mathbb{E}_{AB}^{A \prec B} + p_2 \mathbb{E}_{AB}^{B \prec A} + p_3 \mathbb{E}_{AB}^{B \not\prec \not\succ A}, \tag{C.25}$$

where $\mathbb{E}_{AB}^{A \prec B}$ ($\mathbb{E}_{AB}^{B \prec A}$) denotes a process where Alice (Bob) is in the causal past of Bob (Alice), and $\mathbb{E}_{AB}^{B \not\prec \not\succ A}$ represents a process with A and B being spacelike separated;

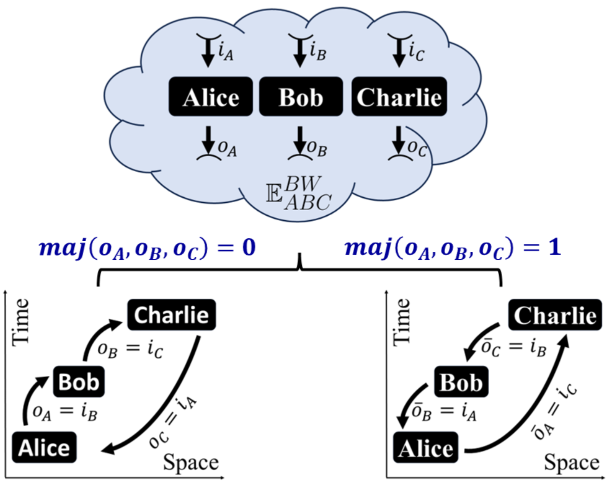

FIG. C.1 The tripartite classical causal indefinite process as given in Eq.(C.26a). While each of the branches, described in Eq.(C.26b), lead to logical paradoxes when described in a definite spacetime, their combination yields a logically-consistent-classical-process $\mathbb{E}^{BW}_{ABC}$.

with $\vec{p} = (p_1, p_2, p_3)^T$ being a probability vector. However, for multipartite case there are classical processes which do not admit a notion of causal ordering among the parties. For instance, consider the classical process,

$$\mathbb{E}^{BW}_{ABC}(o_A, o_B, o_C) = (i_A, i_B, i_C), \text{ with} \tag{C.26a}$$

$$(i_A, i_B, i_C) \equiv \begin{cases} (o_C, o_A, o_B), \\ \qquad \text{if } \text{maj}(o_A, o_B, o_C) = 0, \\ (\bar{o}_B, \bar{o}_C, \bar{o}_A), \\ \qquad \text{if } \text{maj}(o_A, o_B, o_C) = 1, \end{cases} \tag{C.26b}$$

where $o_A, o_B, o_C, i_A, i_B, i_C \in \{0,1\}, \bar{0} = 1$ and $\bar{1} = 0$ (see Fig.C.1). As shown by the the authors in [114], the LCCP $\mathbb{E}^{BW}$ violates a tripartite causal inequality, establishing that causal indefiniteness is no longer an artifact of quantum processes. At this point, one might ask whether advantage in DR task stems from indefinite quantum processes only or is it a general trait of causal indefiniteness. We answer this question in affirmative by providing a tripartite variant of DR task wherein $\mathbb{E}^{BW}$ provides a nontrivial advantage.

C.3.2 T-DR Success Under Different Collaboration Scenarios

In this subsection we analyse the success probability of T-DR task under different collaboration scenarios. We start by considering the no-collaboration case.

Proposition 8. *Without any collaboration Alice, Bob and Charlie can achieve the success* $P_{succ}^{T\text{-}DR} = 27/64$.

Proof. The only way Alice can learn something definitive about her string is through communication from Charlie. Similarly, Bob and Charlie need communication from Alice and Bob, respectively. Without any such communication, the best Alice can do is to answer $a_0 = 0$ and guess a value for $a_1 a_1'$, which leads to a success $3/4$. A similar strategy followed by Bob and Charlie yields an overall success $P_{succ}^{\text{T-DR}} = (3/4)^3 = 27/64$. □

However, unlike the DR-B task, LOCC collaboration turns out to be advantageous in this case.

Proposition 9. *Under LOCC collaboration* $P_{succ}^{T\text{-}DR} = 1$.

Proof. Recall that two-qubit Bell basis shared between two distant parties cannot be perfectly distinguished by LOCC [108]. However, according to the result in [166], given two copies of the states, they can be perfectly distinguished under LOCC. The protocol goes as follows: both the players perform Z-basis measurement on their parts of the first copy and X-basis measurement on second. One of the players communicate the results to the other player, who can accordingly identify the given Bell state. This ensures a perfect success of T-DR task under LOCC. □

However, the protocol in Proposition [9] demands multi-round communication among the players. For instance, let Alice first communicate her results to Bob implying Alice's measurement event to be in the causal past of Bob's guess. Similarly, Bob to Charlie communication demands Bob's measurement event to be in the causal past of Charlie's guess. Finally, Charlie to Alice communication demands Charlie's measurement event to be in the causal past of Alice's guess. Since, in the single-opening setup communication entering into a local laboratory must happen before any communication going outside it, therefore the above protocol cannot be implemented within this setup. Therefore, the question of optimal success of T-DR is worth exploring in single-opening scenario. However, likewise the notions of genuine and non-genuine entanglement on multipartite case [167], the notion of causal indefiniteness can also have different manifestations when more than two parties are involved. Before proceeding further, here we first recall the definition of bi-causal/genuine quantum process.

Proposition 10. *Under bi-causal collaboration* $P_{succ}^{T\text{-}DR} \leq 3/4$.

Proof. In a process of type $W^{A\not\prec BC}$ communication from Alice is not possible to Bob as well as to Charlie. Thus Bob's success is bounded by 3/4 (see Proposition [9]). On the other hand, in a process of type $W^{BC\not\prec A}$ Alice's success is bounded by 3/4 as neither Bob nor Charlie can communicate to Alice. Similar arguments hold for all other terms in Eq.(2.78), and hence the claim follows from convexity. To achieve the bound, they can share a definite order process $W^{A\prec B\prec C}$ where Alice is in the causal past of Bob, who is in the causal past of Charlie. Using the strategy discussed in Proposition [9], Bob's and Charlie's guesses will be perfect whereas Alice's success is bounded by 3/4. This completes the proof. □

Naturally, the question arises whether a genuine inseparable causal process could be advantageous over the bi-causal processes. In the following section, we show this is indeed possible, even with a classical indefinite process.

C.3.3 Nontrivial Success in T-DR with LCCP

Given the $\mathbb{E}^{BW} \in$ LCCP Eq.(C.26), the players can obtain a nontrivial success in T-DR task. In the encoded state given to the players, two Bell states are shared between each pair of the players. Of course, the identity of the Bell state is not known to the individual parties. Given an encoded state, Alice performs Z-basis measurement on her part of one of the the Bell state shared with Bob, and performs X-basis measurement on her part of the the Bell state shared with Bob. Similarly, Z and X measurements are performed on the parts of Bell states shared with Charlie. Bob and Charlie follow a similar protocol. Outcome of all these different measurements can be compactly expressed as $G_K^{(H)} \in \{0,1\}$ – outcome of K-basis measurement performed by the player G on her part of the Bell state shared with the player H; with $K \in \{Z,X\}$ and $G,H \in \{A,B,C\}$. Given the encoded state $\rho_{ABC}^{\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3}$, we have

$$A_Z^{(C)} \oplus C_Z^{(A)} = x_1,\ \ A_X^{(C)} \oplus C_X^{(A)} = x_1', \tag{C.27a}$$

$$B_Z^{(A)} \oplus A_Z^{(B)} = x_2,\ \ B_X^{(A)} \oplus A_X^{(B)} = x_2', \tag{C.27b}$$

$$C_Z^{(B)} \oplus B_Z^{(C)} = x_3,\ \ C_X^{(B)} \oplus B_X^{(C)} = x_3'. \tag{C.27c}$$

For their local measurement outcomes the players respectively evaluate a bit values

$$o_A = A_Z^{(B)} A_X^{(B)},\ \ o_B = B_Z^{(C)} B_X^{(C)},\ \ o_C = C_Z^{(A)} C_X^{(A)}, \tag{C.28}$$

maj	(o_A, o_B, o_C)	(i_A, i_B, i_C)	**a**	**b**	**c**
0	$(0,0,0)$	$(0,0,0)$	$0\neg(00)$	$0\neg(00)$	$0\neg(00)$
	$(0,0,1)$	$(1,0,0)$	100	$0\neg(00)$	$0\neg(00)$
	$(1,0,0)$	$(0,1,0)$	$0\neg(00)$	100	$0\neg(00)$
	$(0,1,0)$	$(0,0,1)$	$0\neg(00)$	$0\neg(00)$	100
1	$(0,1,1)$	$(0,0,1)$	000	$0\neg(00)$	100
	$(1,0,1)$	$(1,0,0)$	100	000	$0\neg(00)$
	$(1,1,0)$	$(0,1,0)$	$0\neg(00)$	100	000
	$(1,1,1)$	$(0,0,0)$	000	000	000

TABLE C.1 Input $\mathbf{x} = \mathbf{0}$. For the case "maj$(o_A, o_B, o_C) = 0$", all three players guess correctly. However, for the case "maj$(o_A, o_B, o_C) = 1$", at-least one of players' guess is not correct (marked in red). Here, $\neg(00)$ indicates any string not equal to 00 i.e. 01/10/11.

and send them to the environment $\mathbb{E}^{BW}$, which on the other hand returns back the bits i_A, i_B, and i_C to Alice, Bob, and Charlie. The guesses in T-DR task for Alice, Bob and Charlie are given by

$$a_0 = i_A, \quad a_1 = \overline{A_Z^{(C)}}, \quad a_1' = \overline{A_X^{(C)}} \tag{C.29a}$$

$$b_0 = i_B, \quad b_1 = \overline{B_Z^{(A)}}, \quad b_1' = \overline{B_X^{(A)}} \tag{C.29b}$$

$$c_0 = i_C, \quad a_1 = \overline{C_Z^{(B)}}, \quad c_1' = \overline{C_X^{(B)}} \tag{C.29c}$$

The success probability for $\mathbf{x} = \mathbf{0} \equiv 000000$, turns out to be

$$\begin{aligned} P_{succ}^{\text{T-DR}}(\mathbf{x} = \mathbf{0}) &= \sum_{o_A, o_B, o_C} \sum_{\mathbf{g} \in \pounds^{\mathbf{0}}} p(o_A o_B o_C \mathbf{g} | \mathbf{x} = \mathbf{0}) \\ = \sum_{maj(o_A, o_B, o_C) = 0} & \sum_{\mathbf{g} \in \pounds^{\mathbf{0}}} p(o_A o_B o_C | \mathbf{x} = \mathbf{0}) p(\mathbf{g} | \mathbf{x} = \mathbf{0} o_A o_B o_C) + \\ \sum_{maj(o_A, o_B, o_C) = 1} & \sum_{\mathbf{g} \in \pounds^{\mathbf{0}}} p(o_A o_B o_C | \mathbf{x} = \mathbf{0}) p(\mathbf{g} | \mathbf{x} = \mathbf{0} o_A o_B o_C). \end{aligned} \tag{C.30}$$

As we can see from Table C.1, for the case "maj$(o_A, o_B, o_C) = 1$", atleast one of the players violates the winning condition (5.14), *i.e.*,

$$\sum_{\mathbf{g} \in \pounds^{\mathbf{0}}} p(\mathbf{g} | \mathbf{x} = \mathbf{0} o_A o_B o_C) = 0. \tag{C.31}$$

However, for the case “maj$(o_A, o_B, o_C) = 0$", all the players satisfy the winning condition (5.14), *i.e.*,

$$\sum_{\mathbf{g}\in\pounds^{\mathbf{0}}} p(\mathbf{g}|\mathbf{x} = \mathbf{0}o_A o_B o_C) = 1. \tag{C.32}$$

Consequently, Eq.(C.30) becomes

$$\begin{aligned} P^{\text{T-DR}}_{succ}(\mathbf{x} = \mathbf{0}) &= \sum_{maj(o_A,o_B,o_C)=0} p(o_A o_B o_C|\mathbf{x} = \mathbf{0}) \\ &= \left[\frac{3^3}{4^3} + 3\times\frac{3^2}{4^3}\right] = \frac{27}{32} \approx 0.84 > \frac{3}{4}\ . \end{aligned} \tag{C.33}$$

Similarly, it can be shown that $P^{\text{T-DR}}_{succ}(\mathbf{x}) = 27/32,\ \forall\ \mathbf{x} \in \{0,1\}^{\times 6}$, leading to $P^{\text{T-DR}}_{succ} = 27/32$. Therefore, the classical causally indefinite process $\mathbb{E}^{EW}$ exhibits non-trivial advantage over the quantum bi-causal processes in T-DR task. In fact, the success establishes the genuine multipartite nature of causal indefiniteness. Note that in the above mentioned protocol all the players are efficiently able to communicate the required information by effectively implementing the maj$(o_A, o_B, o_C) = 0$ loop in Fig.(C.1) with a high probability. One can say that effectively clockwise communication is happening between the players. This clockwise communication is also in some sense necessary, as the encoding states also have this "clockwise" property (see Eq.(5.11)) i.e. Alice needs help from Charlie, Bob needs help from Alice and Charlie needs help from Bob . In Appendix [C.3.4] we discuss an interesting variant of the T-DR task where the referee does not reveal whether they have done a clockwise or anticlockwise encoding but rather encodes this information in the distributed state itself. Interestingly, we show that even though the three players beforehand do not know whether the referee has encoded in a clockwise or anticlockwise fashion the process $\mathbb{E}^{BW}$ still can provide an advantage by effectively using both branches in Fig.(C.1) which is impossible to do by a definite ordered process.

C.3.4 Flagged T-DR

In this flagged version of T-DR task (FT-DR) referee encodes the strings $\mathbf{x} = \mathbf{x_1 x_2 x_3}$ into

$$\begin{aligned}\rho^{\mathbf{x}}_{AA'BB'CC'} = &\frac{1}{2}\big[|000\rangle\langle 000|_{A'B'C'} \\ &\quad \otimes(\mathscr{B}^{\mathbf{x_1}}_{AC})^{\otimes 2}\otimes(\mathscr{B}^{\mathbf{x_2}}_{BA})^{\otimes 2}\otimes(\mathscr{B}^{\mathbf{x_3}}_{CB})^{\otimes 2}\big] \\ &+\frac{1}{2}\big[|111\rangle\langle 111|_{A'B'C'} \\ &\quad \otimes(\mathscr{B}^{\mathbf{x_1}}_{AB})^{\otimes 2}\otimes(\mathscr{B}^{\mathbf{x_2}}_{BC})^{\otimes 2}\otimes(\mathscr{B}^{\mathbf{x_3}}_{CA})^{\otimes 2}\big]. \end{aligned} \tag{C.34}$$

Winning condition for FT-DR remains same as of Eq.(5.14). All the players perform Z basis measurement on the flagged state (primed systems). If they obtain outcome '0', they follow the strategy of T-DR with $\mathbb{E}^{BW}$. Otherwise, Eqs.(C.27) get modified as

$$A^{(B)}_Z \oplus B^{(A)}_Z = x_1,\;\; A^{(B)}_X \oplus B^{(A)}_X = x'_1, \tag{C.35a}$$
$$B^{(C)}_Z \oplus C^{(B)}_Z = x_2,\;\; B^{(C)}_X \oplus C^{(B)}_X = x'_2, \tag{C.35b}$$
$$C^{(A)}_Z \oplus A^{(C)}_Z = x_3,\;\; C^{(A)}_X \oplus A^{(C)}_X = x'_3. \tag{C.35c}$$

In this case the players encode as

$$\bar{o}_A = A^{(C)}_Z A^{(C)}_X,\; \bar{o}_B = B^{(A)}_Z B^{(A)}_X,\; \bar{o}_C = C^{(B)}_Z C^{(B)}_X. \tag{C.36}$$

And their guesses are

$$a_0 = i_A,\; a_1 = \overline{A^{(B)}_Z},\; a'_1 = \overline{A^{(B)}_X} \tag{C.37a}$$
$$b_0 = i_B,\; b_1 = \overline{B^{(C)}_Z},\; b'_1 = \overline{B^{(C)}_X} \tag{C.37b}$$
$$c_0 = i_C,\; a_1 = \overline{C^{(A)}_Z},\; c'_1 = \overline{C^{(A)}_X} \tag{C.37c}$$

From symmetry of $\mathbb{E}^{BW}$, it follows that $P^{\text{FT-DR}}_{succ} \approx 0.84$. While $P^{\text{T-DR}}_{succ} = 3/4$ can be achieved in definite causal structure, it is not the case for FT-DR task. To see this consider the case $A \prec B \prec C$.
(i) if outcome on flagged state is '0', then they can ensure a success $3/4$: Alice and Bob can respectively help Bob and Charlie to guess their respective messages correctly.
(ii) for '1' outcome on flagged state, a success of $3^2/4^2$ can be ensured. While Alice can help Charlie only, Alice and Bob have to guess their respective messages.

Thus on an average the success becomes

$$P_{succ}^{\text{FT-DR}} = \frac{1}{2}\left(\frac{3}{4} + \frac{3^2}{4^2}\right) = \frac{21}{32} < \frac{3}{4}. \tag{C.38}$$

This demonstrates that sharing $\mathbb{E}^{BW}$ allows the players to effectively communicate in clockwise or anticlockwise fashion by suitably modifying their protocols of T-DR task on their will without giving rise to casual loops. However any causally ordered process would fail miserably to do so. Like Proposition [10], in this case too obtaining a nontrivial bound for bi-causal quantum processes is not straightforward.

Appendix D

Supplementary Material for Chapter [6]

D.1 More on the proof of Theorem [18]

We now provide a more detailed and rigorous proof of Theorem [18]. Consider the expression for effective POVM F_ψ:

$$\mathrm{F}_\psi = \frac{1}{2}\mathrm{P}_{\hat{\psi}^\perp} = \sum_m \int dx\; p(x)\; p(m|x,\psi)\; \mathrm{E}^{m,x}. \tag{D.1}$$

Let us assume that M is the cardinality of the message set $\{m\}_{m=0}^{M-1}$. Notably, without loss of any generality we can assume $p(x) > 0$ for all x. Let us now define the set

$$\mathbf{P} := \{(m,x) \mid p(m|x,\psi) > 0 \text{ for some } \psi\}. \tag{D.2}$$

Since the left-hand side of Eq. (D.1) is a rank-1 operator for all ψ, it follows that $\mathrm{E}^{m,x}$ must have rank at most 1 and can be expressed as $\mathrm{E}^{m,x} = e^{m,x} |\chi^{m,x}\rangle \langle\chi^{m,x}|$, where $0 \leq e^{m,x} \leq 1$ for all $(m,x) \in \mathbf{P}$. Furthermore, we can assume that $\mathrm{E}^{m,x} = e^{m,x} |\chi^{m,x}\rangle \langle\chi^{m,x}|$ holds even for $(m,x) \notin \mathbf{P}$, since such terms do not contribute to the right-hand side of Eq. (D.1) for any ψ. Substituting this decomposition into Eq. (D.1), we obtain

$$\frac{1}{2}\mathrm{P}_{\hat{\psi}^\perp} = \sum_m \int dx\; p(x)\; p(m|x,\psi)\; e^{m,x} |\chi^{m,x}\rangle \langle\chi^{m,x}|. \tag{D.3}$$

Next, we define the set $\Lambda_\psi^m := \{x \mid p(m|x,\psi) e^{m,x} > 0\}$. Thus, we can rewrite Eq. (D.1) as

$$\frac{1}{2}\mathrm{P}_{\hat{\psi}^\perp} = \sum_m \int_{\Lambda_\psi^m} dx\; p(x)\; p(m|x,\psi)\; e^{m,x} |\chi^{m,x}\rangle \langle\chi^{m,x}|. \tag{D.4}$$

Since the left-hand side of Eq. (D.4) is proportional to $P_{\hat{\psi}^\perp}$, it follows that $|\chi^{m,x}\rangle\langle\chi^{m,x}| = P_{\hat{\psi}^\perp}$, $\forall\, x \in \Lambda_\psi^m$. Thus, we must also have $\Lambda_\psi^m \cap \Lambda_{\psi'}^m = \emptyset$, $\forall\, m$ & $\psi \neq \psi'$. Taking the trace on both sides of Eq. (D.4), we obtain

$$\frac{1}{2} = \sum_m \int_{\Lambda_\psi^m} dx\; p(x)\; p(m|x,\psi)\; e^{m,x} \leq \sum_m \int_{\Lambda_\psi^m} dx\; p(x) \leq M \times \max_m \left\{ \int_{\Lambda_\psi^m} dx\; p(x) \right\}, \;\forall\; \psi. \tag{D.5}$$

Since there are uncountably many values of ψ and only finitely many values of m, by the pigeonhole principle, there must exist some m_0 such that $\int_{\Lambda_\psi^m} dx\; p(x)$ attains its maximum for $m = m_0$ for uncountably many values of ψ. Consequently, we must have

$$\frac{1}{2M} \leq \int_{\Lambda_\psi^{m_0}} dx\; p(x), \tag{D.6}$$

for uncountably many values of ψ, while also satisfying the disjoint-ness condition:

$$\Lambda_\psi^{m_0} \cap \Lambda_{\psi'}^{m_0} = \emptyset, \quad \forall\; \psi \neq \psi'. \tag{D.7}$$

However, conditions (D.6) and (D.7) are impossible to satisfy for any finite M, as the total probability must satisfy $\int dx\; p(x) = 1$. Thus, we arrive at a contradiction, proving that it is impossible to simulate a qubit with finite classical communication in the given scenario.

D.2 Proof of Proposition [7]

We first start by proving any 3 round protocol with finite communication in every round can be implemented by a 1 way finite communication protocol

D.2.1 3-way communication protocol

Let x be the shared variable between Alice and Bob. Alice starts by tossing a coin $q(m_1|\psi x)$ and communicating m_1 to Bob. Bob performs a measurement $M_{m_1 x} \equiv \{M_{m_1 x}^{m_2}\}$. Here writing general measurement operators is necessary, as POVM elements do not uniquely specify the post measurement state, which is essential for knowing statistics for further measurements. We have $\sum_{m_2} M_{m_1 x}^{m_2 \dagger} M_{m_1 x}^{m_2} = \mathbb{I}\;\forall\; m_1, x$. Next Bob communicated his outcome m_2 back to Alice and then Alice tosses the coin $r(m_3|m_1 m_2 \psi x)$ and communicates m_3. Bob then performs a POVM measurement $M_{m_1 m_2 m_3 x} = \{\pi^b_{m_1 m_2 m_3 x}\}$ on the updated state.

Here $\sum_b \pi^b_{m_1m_2m_3x} = \mathbb{I}\ \forall\ m_1, m_2, m_3, x$. Mathematically the correlation $p(b|\psi\phi)$ can be written as follows:

$$\begin{aligned}
p(b|\psi\phi) &= \sum_{m_1m_2m_3x} p(bm_1m_2m_3x|\psi\phi) = \sum_{m_1m_2m_3x} p(x|\psi\phi)p(bm_1m_2m_3|\psi\phi x) \\
&= \sum_{m_1m_2m_3x} \mu(x)p(m_1|\psi\phi x)p(bm_2m_3|m_1\psi\phi x) \\
&= \sum_{m_1m_2m_3x} \mu(x)q(m_1|\psi x)p(m_2|m_1\psi\phi x)p(bm_3|m_1m_2\psi\phi x) \\
&= \sum_{m_1m_2m_3x} \mu(x)q(m_1|\psi x)\ \mathrm{Tr}[M^{m_2}_{m_1x}\phi M^{m_2\dagger}_{m_1x}]\ p(bm_3|m_1m_2\psi\phi x) \\
&= \sum_{m_1m_2m_3x} \mu(x)q(m_1|\psi x)\ \mathrm{Tr}[M^{m_2}_{m_1x}\phi M^{m_2\dagger}_{m_1x}]\ p(m_3|m_1m_2\psi\phi x)p(b|m_1m_2m_3\psi\phi x) \\
&= \sum_{m_1m_2m_3x} \mu(x)q(m_1|\psi x)\ \mathrm{Tr}[M^{m_2}_{m_1x}\phi M^{m_2\dagger}_{m_1x}]\ p(m_3|m_1m_2\psi\phi x)p(b|m_1m_2m_3\psi\phi x) \\
&= \sum_{m_1m_2m_3x} \mu(x)q(m_1|\psi x)\ \mathrm{Tr}[M^{m_2}_{m_1x}\phi M^{m_2\dagger}_{m_1x}]\ r(m_3|m_1m_2\psi x)p(b|m_1m_2m_3\psi\phi x) \\
&= \sum_{m_1m_2m_3x} \mu(x)q(m_1|\psi x)\ \mathrm{Tr}[M^{m_2}_{m_1x}\phi M^{m_2\dagger}_{m_1x}]\ r(m_3|m_1m_2\psi x)\times \\
&\qquad\qquad \left\{\mathrm{Tr}\left[\pi^b_{m_1m_2m_3x}\left(\frac{M^{m_2}_{m_1x}\phi M^{m_2\dagger}_{m_1x}}{\mathrm{Tr}[M^{m_2}_{m_1x}\phi M^{m_2\dagger}_{m_1x}]}\right)\right]\right\} \\
&= \sum_{m_1m_2m_3x} \mu(x)q(m_1|\psi x)r(m_3|m_1m_2\psi x)\left\{\mathrm{Tr}\left[\pi^b_{m_1m_2m_3x}\left(M^{m_2}_{m_1x}\phi M^{m_2\dagger}_{m_1x}\right)\right]\right\}
\end{aligned} \tag{D.8a}$$

In the next subsection we show that the above correlation can be simulated by finite 1 way protocol.

D.2.2 1-way communication protocol to simulate the 3-way protocol in D.2.1

Here Alice tosses the following coins $q(m_1|\psi x), r(\eta_1|\psi x m_1, m_2 = 1), r(\eta_2|\psi x m_1, m_2 = 2), \cdots, r(\eta_k|\psi x m_1, m_2 = k)$. Here we have assumed m_2 takes values in the set $\{1, \cdots, k\}$. Alice then communicates the outcomes $m_1, \eta_1, \cdots, \eta_k$. Bob follows the same protocol for first measurement and gets an outcome m_2 and then he generates $m_3 = \eta_{m_2}$ and then follows the same protocol as in the 3 way case. Here we aim to prove that this particular protocol yields the same $p(b|\psi\phi)$ as in

Eq.(D.8a).

$$p(b|\psi\phi) = \sum_{m_1 x} p(bm_1 x|\psi\phi) = \sum_{m_1 x} \mu(x) q(m_1|\psi x) p(b|m_1 \psi\phi x) \tag{D.9a}$$

Now we write $p(b|m_1\psi\phi x)$ as

$$\begin{aligned} p(b|m_1\psi\phi x) &= \sum_{\eta_1\cdots\eta_k} p(b\eta_1\cdots\eta_k|m_1\psi\phi x) = \sum_{\eta_1\cdots\eta_k} p(\eta_1|m_1\psi\phi x) p(b\eta_2\cdots\eta_k|m_1\psi\phi x\eta_1) \\ &= \sum_{\eta_1\cdots\eta_k} r(\eta_1|\psi x m_1, m_2 = 1) p(\eta_2|m_1\psi\phi x\eta_1) p(b\eta_3\cdots\eta_k|m_1\psi\phi x\eta_1\eta_2) \\ &= \sum_{\eta_1\cdots\eta_k} r(\eta_1|\psi x m_1, m_2 = 1) r(\eta_2|\psi x m_1, m_2 = 2) p(b\eta_3\cdots\eta_k|m_1\psi\phi x\eta_1\eta_2) \end{aligned}$$

Repeating the same procedure we get

$$p(b|m_1\psi\phi x) = \sum_{\eta_1\cdots\eta_k} \left\{ \prod_{i=1}^{k} r(\eta_i|\psi x m_1, m_2 = i) \right\} p(b|m_1\phi x\eta_1\cdots\eta_k) \tag{D.10a}$$

Now we compute $p(b|m_1\phi x\eta_1\cdots\eta_k)$

$$\begin{aligned} p(b|m_1\phi x\eta_1\cdots\eta_k) &= \sum_{m_2} p(m_2 b|m_1\phi x\eta_1\cdots\eta_k) \\ &= \sum_{m_2} p(m_2|m_1\phi x\eta_1\cdots\eta_k) p(b|m_1 m_2\phi x\eta_1\cdots\eta_k) \\ &= \sum_{m_2} p(m_2|m_1\phi x) p(b|m_1 m_2\phi x\eta_1\cdots\eta_k) \\ &= \sum_{m_2} \mathrm{Tr}[M^{m_2}_{m_1 x}\phi M^{m_2\dagger}_{m_1 x}]\ p(b|m_1 m_2\phi x\eta_1\cdots\eta_k) \\ &= \sum_{m_2} \mathrm{Tr}[M^{m_2}_{m_1 x}\phi M^{m_2\dagger}_{m_1 x}]\ p(b|m_1 m_2\phi x\eta_{m_2}) \\ &= \sum_{m_2} \mathrm{Tr}[M^{m_2}_{m_1 x}\phi M^{m_2\dagger}_{m_1 x}]\ \left\{ \mathrm{Tr}\left[\pi^b_{m_1 m_2 \eta_{m_2} x} \left(\frac{M^{m_2}_{m_1 x}\phi M^{m_2\dagger}_{m_1 x}}{\mathrm{Tr}[M^{m_2}_{m_1 x}\phi M^{m_2\dagger}_{m_1 x}]} \right) \right] \right\} \\ &= \sum_{m_2} \mathrm{Tr}\left[\pi^b_{m_1 m_2 \eta_{m_2} x} \left(M^{m_2}_{m_1 x}\phi M^{m_2\dagger}_{m_1 x} \right) \right] \end{aligned} \tag{D.11a}$$

Replacing m_2 by another dummy variable m_2' in Eq.(D.11a) and substituting $p(b|m_1\phi x\eta_1\cdots\eta_k)$ in Eq.(D.10a) we get

$$\begin{aligned}
p(b|m_1\psi\phi x) &= \sum_{\eta_1\cdots\eta_k}\left\{\prod_{i=1}^{k} r(\eta_i|\psi x m_1, m_2=i)\right\}\left(\sum_{m_2'}\mathrm{Tr}\left[\pi^b_{m_1 m_2' \eta_{m_2'} x}\left(M^{m_2'}_{m_1 x}\phi M^{m_2'\dagger}_{m_1 x}\right)\right]\right)\\
&=\sum_{m_2'}\sum_{\eta_1\cdots\eta_k}\left\{\prod_{i=1}^{k} r(\eta_i|\psi x m_1, m_2=i)\,\mathrm{Tr}\left[\pi^b_{m_1 m_2' \eta_{m_2'} x}\left(M^{m_2'}_{m_1 x}\phi M^{m_2'\dagger}_{m_1 x}\right)\right]\right\}\\
&=\sum_{m_2'}\sum_{\eta_1\cdots\eta_k}\left\{\left(\prod_{i\neq m_2'} r(\eta_i|\psi x m_1, m_2=i)\right)\ r(\eta_{m_2'}|\psi x m_1, m_2=m_2')\times\right.\\
&\qquad\left.\mathrm{Tr}\left[\pi^b_{m_1 m_2' \eta_{m_2'} x}\left(M^{m_2'}_{m_1 x}\phi M^{m_2'\dagger}_{m_1 x}\right)\right]\right\}\\
&=\sum_{m_2'}\sum_{\eta_{m_2'}}\left\{\left(\prod_{i\neq m_2'}\sum_{\eta_i} r(\eta_i|\psi x m_1, m_2=i)\right)\ r(\eta_{m_2'}|\psi x m_1, m_2=m_2')\times\right.\\
&\qquad\left.\mathrm{Tr}\left[\pi^b_{m_1 m_2' \eta_{m_2'} x}\left(M^{m_2'}_{m_1 x}\phi M^{m_2'\dagger}_{m_1 x}\right)\right]\right\}.
\end{aligned}$$

Since $\sum_{\eta_i} r(\eta_i|\psi x m_1, m_2=i)=1$ and $\prod_{i\neq m_2'} 1 = 1$ we have

$$p(b|m_1\psi\phi x) = \sum_{m_2'\eta_{m_2'}} r(\eta_{m_2'}|\psi x m_1, m_2=m_2')\ \mathrm{Tr}\left[\pi^b_{m_1 m_2' \eta_{m_2'} x}\left(M^{m_2'}_{m_1 x}\phi M^{m_2'\dagger}_{m_1 x}\right)\right]$$

$$p(b|m_1\psi\phi x) = \sum_{m_2 m_3} r(m_3|\psi x m_1 m_2)\ \mathrm{Tr}\left[\pi^b_{m_1 m_2 m_3 x}\left(M^{m_2}_{m_1 x}\phi M^{m_2\dagger}_{m_1 x}\right)\right] \tag{D.12a}$$

Where we replaced the dummy variables m_2' and $\eta_{m_2'}$ by the variables m_2 and m_3 respectively. Now substituting $p(b|m_1\psi\phi x)$ from Eq.(D.12a) in Eq.(D.9a) we get,

$$p(b|\psi\phi) = \sum_{m_1 m_2 m_3 x} \mu(x) q(m_1|\psi x)\ r(m_3|\psi x m_1 m_2)\ \mathrm{Tr}\left[\pi^b_{m_1 m_2 m_3 x}\left(M^{m_2}_{m_1 x}\phi M^{m_2\dagger}_{m_1 x}\right)\right] \tag{D.13}$$

Which exactly matches the expression for the 3 way case. This proves that any 3 way bounded communication protocol can be simulated by a 1 way bounded communication protocol. This also proves that we can also convert any bounded back and forth communication to only 1 way communication.

D.3 Proof of Observation [1]

Proof. We start by recalling the $2 \mapsto 1$ RAC task [42, 155], where Alice is provided with a random bit string $x_0x_1 \in \{0,1\}^2$ and Bob is randomly given $y \in \{0,1\}$. Bob's aim to produce a 1-bit outcome $b = x_y$ with the help of 1-bit respectively 1-qubit communication from Alice. Qubit strategies yield the optimal success $P_Q = 1/2(1+1/\sqrt{2})$ which is strictly higher than the optimal c-bit success $P_C = 1/2(1+1/2)$.

Contrary to the claim of the Observation 1, let us assume that the statistics of the measurement $\mathbf{M}^{(A)}_{twist} \equiv \{\mathrm{P}_{\hat{z}} \otimes \mathrm{P}_{\hat{z}}, \mathrm{P}_{\hat{z}^\perp} \otimes \mathrm{P}_{\hat{z}}, \mathrm{P}_{\hat{x}} \otimes \mathrm{P}_{\hat{z}^\perp}, \mathrm{P}_{\hat{x}^\perp} \otimes \mathrm{P}_{\hat{z}^\perp}\}$ on a state known to Alice and an unknown state of Bob system can be simulated at Bob's end with just 1-bit of classical communication from Alice to Bob. Let us denote this protocol as 1-CBS. As we will argue now, this protocol can be utilized to perform the $2 \mapsto 1$ RAC task. Given the bit string Alice will implement the 1-CBS protocol on the preparation

$$\psi_A^{x_0x_1} = \frac{1}{2}\left[\mathbf{I}_2 + \frac{1}{\sqrt{2}}\{(-1)^{x_0}\sigma_3 + (-1)^{x_1}\sigma_1\}\right], \tag{D.14}$$

whereas Bob, given the question y, will prepare the state

$$\phi_B^y = \frac{1}{2}\left[\mathbf{I}_2 + (-1)^y\sigma_3\right]. \tag{D.15}$$

As per the assumption, 1-CBS protocol reproduce the statistics of $\mathbf{M}^{(A)}_{twist}$ on $\psi_A^{x_0x_1} \otimes \phi_B^y$ at Bob's laboratory. Bob can post-process this outcome statistics and can accordingly devise a strategy to answer his guess b. In particular, for the outcomes $\mathrm{P}_{\hat{z}} \otimes \mathrm{P}_{\hat{z}}$ and $\mathrm{P}_{\hat{x}} \otimes \mathrm{P}_{\hat{z}^\perp}$ Bob guesses $b=0$, else he guesses $b=1$. Denoting $\Pi_0 = \mathrm{P}_{\hat{z}} \otimes \mathrm{P}_{\hat{z}} + \mathrm{P}_{\hat{x}} \otimes \mathrm{P}_{\hat{z}^\perp}$ and $\Pi_1 = \mathrm{P}_{\hat{z}^\perp} \otimes \mathrm{P}_{\hat{z}} + \mathrm{P}_{\hat{x}^\perp} \otimes \mathrm{P}_{\hat{z}^\perp}$, we have

$$\begin{aligned}\mathbf{Pr}(b = x_y | x_0x_1, y) &= \mathrm{Tr}\left[\left(\psi_A^{x_0x_1} \otimes \phi_B^y\right)\Pi_{b=x_y}\right] \\ &= \frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right), \ \forall\ x_ox_1\ \&\ y.\end{aligned} \tag{D.16}$$

Therefore, with 1-CBS protocol one can have a success $P_{1-\mathrm{CBS}} = 1/2(1+1/\sqrt{2})$ in $2 \mapsto 1$ RAC task – a contradiction. In other words, this proves that with 1-bit communication the statistics of $\mathbf{M}^{(A)}_{twist}$ cannot be reproduced at Bob's laboratory. □

D.4 An Explicit Example of a Nontrivial Product Von-Neumann Measurement simulable by finite classical communication

For a better clarification of Theorem [20], here we provide an explicit example. Consider the OPB $\mathbf{B} = \mathbf{B}_1 \cup \mathbf{B}_2 \cup \mathbf{B}_3$ of $\mathbb{C}^2 \otimes \mathbb{C}^6$ system, where

$$\mathbf{B}_1 := \left\{ |0\rangle|0\rangle, |0\rangle|1\rangle, |1\rangle|x_+^{01}\rangle, |1\rangle|x_-^{01}\rangle \right\}, \tag{D.17a}$$

$$\mathbf{B}_2 := \left\{ |x_+^{01}\rangle|2\rangle, |x_+^{01}\rangle|3\rangle, |x_-^{01}\rangle|x_+^{23}\rangle, |x_-^{01}\rangle|x_-^{23}\rangle \right\}, \tag{D.17b}$$

$$\mathbf{B}_2 := \left\{ |y_+^{01}\rangle|4\rangle, |y_+^{01}\rangle|5\rangle, |y_-^{01}\rangle|x_+^{45}\rangle, |y_-^{01}\rangle|x_-^{45}\rangle \right\}, \tag{D.17c}$$

with, $|x_\pm^{lm}\rangle := \frac{1}{\sqrt{2}}(|l\rangle \pm |m\rangle)$ and $|y_\pm^{lm}\rangle := \frac{1}{\sqrt{2}}(|l\rangle \pm i|m\rangle)$. Here, i denotes the square root of -1. To simulate statistics of the measurement on this basis, Bob, on his unknown state, first performs a measurement $\mathbf{M}_{1^{st}}^B$ consisting of three rank-2 projective effects

$$\left\{ \begin{array}{c} \mathbb{P}_1 := |0\rangle\langle 0| + |1\rangle\langle 1|,\ \mathbb{P}_2 := |2\rangle\langle 2| + |3\rangle\langle 3|, \\ \mathbb{P}_3 := |4\rangle\langle 4| + |5\rangle\langle 5| \end{array} \right\}. \tag{D.18}$$

On the other hand, Alice performs σ_z, σ_x, and σ_y measurements on three copies of her known state, and communicates the outcomes through three 1-bit classical channels, respectively 1^{st}, 2^{nd}, and 3^{rd}, to Bob. Depending on which projector clicks in his first measurement, Bob chooses the corresponding communication line from Alice, and depending on the communication received from Alice, he performs the measurements as shown is Table [D.1]. This protocol exactly re-

Outcome of $\mathbf{M}_{1st}^B$	Selected Channel	Communication	Bob's final measurement
$\mathbb{P}_1$	1^{st} Channel	0	$\{\|0\rangle\langle 0\|, \|1\rangle\langle 1\|\}$
		1	$\{\|x_+^{01}\rangle\langle x_+^{01}\|, \|x_-^{01}\rangle\langle x_-^{01}\|\}$
$\mathbb{P}_2$	2^{nd} Channel	0	$\{\|2\rangle\langle 2\|, \|3\rangle\langle 3\|\}$
		1	$\{\|x_+^{23}\rangle\langle x_+^{23}\|, \|x_-^{23}\rangle\langle x_-^{23}\|\}$
$\mathbb{P}_3$	3^{rd} Channel	0	$\{\|4\rangle\langle 4\|, \|5\rangle\langle 5\|\}$
		1	$\{\|x_+^{45}\rangle\langle x_+^{45}\|, \|x_-^{45}\rangle\langle x_-^{45}\|\}$

TABLE D.1 Bob selects the i^{th} communication line if rank-2 projector $\mathbb{P}_i$ clicks in his first measurement. Then based on the communication received from Alice through the respective classical channel, he chooses his final measurement.

produces the measurement statistics at Bob's end, while utilizing three classical bits from Alice.

D.5 Simulation of Twisted Butterfly Measurement with 2 bit communication

The twisted-Butterfly POVM M_{tb} induces the following effective POVM on Bob's part:

$$\mathrm{M}^e_{tb} \equiv \left\{ \begin{array}{c} \Pi^e_1 := \frac{1}{2}(1+\psi_z)\mathrm{P}_{\hat{z}^\perp},\ \ \Pi^e_{21} := \frac{3}{8}(1-\frac{2\sqrt{2}}{3}\psi_x+\frac{1}{3}\psi_z)\mathrm{P}_{\hat{z}}, \\ \Pi^e_{22} := \frac{3}{8},(1-\psi_z)\mathrm{P}_{\hat{\alpha}},\ \ \Pi^e_{31} := \frac{3}{8}(1-\frac{2\sqrt{2}}{3}\psi_x+\frac{1}{3}\psi_z)\mathrm{P}_{\hat{z}}, \\ \Pi^e_{32} := \frac{3}{8}(1-\psi_z)\mathrm{P}_{\hat{\beta}^\perp} \end{array} \right\}, \tag{D.19}$$

where ψ_z denotes the z component of the Bloch vector of of Alice's known state $|\psi\rangle$. Using the four projectors $\left\{\mathrm{P}_{\hat{z}}, \mathrm{P}_{\hat{z}^\perp}, \mathrm{P}_{\hat{\alpha}}, \mathrm{P}_{\hat{\beta}^\perp}\right\}$ one can obtain only four rank-1 extremal POVMs, namely

$$\left\{ \begin{array}{ll} \mathrm{M}^1 := \left\{\mathrm{P}_{\hat{z}^\perp}, \mathrm{P}_{\hat{z}}, 0,0,0\right\}, & \mathrm{M}^2 := \left\{\mathrm{P}_{\hat{z}^\perp}, 0,0,\mathrm{P}_{\hat{z}}, 0\right\}, \\ \mathrm{M}^3 := \left\{0, \frac{1}{2}\mathrm{P}_{\hat{z}}, \frac{3}{4}\mathrm{P}_{\hat{\alpha}}, 0, \frac{3}{4}\mathrm{P}_{\hat{\beta}^\perp}\right\}, & \mathrm{M}^4 := \left\{0,0,\frac{3}{4}\mathrm{P}_{\hat{\alpha}}, \frac{1}{2}\mathrm{P}_{\hat{z}}, \frac{3}{4}\mathrm{P}_{\hat{\beta}^\perp}\right\} \end{array} \right\}, \tag{D.20}$$

The POVM M^e_{tb} allows a convex decomposition in terms of extremal POVMs $\{\mathrm{M}^\lambda\}_{\lambda=1}^4$, i.e.

$$\mathrm{M}^e_{tb} = \sum_{i=1}^4 \mu_i(\psi)\mathrm{M}^i, \tag{D.21}$$

where,

$$\left. \begin{array}{ll} \mu_1(\psi) = \max\{0, \frac{1}{8}(1-2\sqrt{2}\psi_x+3\psi_z)\}, & \mu_2(\psi) = \frac{1}{2}(1+\psi_z)-\mu_1(\psi), \\ \mu_3(\psi) = \frac{3}{4}(1-\frac{2\sqrt{2}}{3}\psi_x+\frac{1}{3}\psi_z)-2\mu_1(\psi), & \mu_4(\psi) = \frac{3}{4}(1+\frac{2\sqrt{2}}{3}\psi_x+\frac{1}{3}\psi_z)-2\mu_2(\psi) \end{array} \right\}. \tag{D.22}$$

It is easy to verify that $\{\mu_\lambda(\psi)\}_{\lambda=1}^4$ is indeed a probability distribution $\forall\ \psi$. To simulate statistics of twisted-butterfly POVM, Alice after receiving classical description of the state ψ communicates a four valued random variable

$\{\lambda\}_{\lambda=1}^4$ sampled according to a distribution $\{\mu_\lambda(\psi)\}_{\lambda=1}^4$, and then Bob accordingly performs the measurement M^λ on his unknown state ϕ. Thus 2 bits of communication channel is required from Alice to Bob to implement the classical simulation protocol.

Remark 1. *While the measurement $\mathrm{M}_{twist}^{(A)}$ is LOCC-implementable, the measurement M_{tb} is not implementable via LOCC (Lemma [3]). However, simulation of the outcome statistics on Bob's end for both the measurements is not possible with 1 bit classical communication from Alice, but possible with 2 bits of communication.*

D.6 Proof of Theorem [21]

We start by recalling a definition from [168] (see Section **2.3.3** in page **113**).

Definition 38. *[Rank-1 extremal POVM] A k outcome POVM $\mathbf{M} \equiv \{\Pi_a\}_{a=1}^k$ is called rank-1 extremal POVM if for all a, $\Pi_a = p_a \mathrm{P}_a$, with $p_a \geq 0$ & P_a being a rank-1 projector, and $\sum_a r_a \mathrm{P}_a = 0$ implies $r_a p_a = 0\ \forall\ a$; or equivalently all nonzero elements in $\{\Pi_a\}_{a=1}^k$ are linearly independent of each other. Let, $\mathbf{M}_{R1}^{ext}$ denotes the set of all rank-1 extremal POVMs.*

The notion of rank-1 extremal POVMs leads us to the following useful Lemma.

Lemma 5. *Any finite outcome rank-1 POVM $\mathbf{M}_{R1} \equiv \{s_a \mathrm{P}_a\}_{a=1}^k$ can be written as probabilistic mixture of finite number of rank-1 extremal POVMs, i.e., $\forall\ a$, $s_a \mathrm{P}_a = \sum_{\lambda=1}^{L<\infty} \mu_\lambda s_a^\lambda \mathrm{P}_a$, with $\sum_{\lambda=1}^L \mu_\lambda = 1$ and $\forall\ \lambda$, $\mathbf{M}^\lambda \equiv \{s_a^\lambda \mathrm{P}_a\}_{a=1}^k \in \mathbf{M}_{R1}^{ext}$.*

Proof. Consider an arbitrary rank-1 POVM with finite outcomes $\mathbf{M}_{R1} \equiv \{s_a \mathrm{P}_a\}_{a=1}^k$, with $s_a \geq 0$. According to Definition 38, $\mathbf{M}_{R1}$ allows convex decomposition in terms of $\mathbf{M}^\lambda \equiv \{s_a^\lambda \mathrm{P}_a^\lambda\}_{a=1}^k \in \mathbf{M}_{RI}^{ext}$, i.e.

$$s_a \mathrm{P}_a = \int_\lambda d\lambda \mu_\lambda s_a^\lambda \mathrm{P}_a^\lambda, \text{ with } \mu_\lambda > 0 \ \ \& \int_\lambda d\lambda \mu_\lambda = 1. \tag{D.23}$$

P_a being a rank-1 projector it follows that $\mathrm{P}_a^\lambda = \mathrm{P}_a$, whenever $s_a^\lambda > 0$. On the other hand, for $s_a^\lambda = 0$ also we can assume $\mathrm{P}_a^\lambda = \mathrm{P}_a$, which thus implies $\mathrm{P}_a^\lambda = \mathrm{P}_a,\ \forall\ a, \lambda$. Thus we have $\mathbf{M}^\lambda \equiv \{s_a^\lambda \mathrm{P}_a\}_{a=1}^k$. For such an $\mathbf{M}^\lambda$ we can define $\mathscr{A}_\lambda := \{a \mid s_a^\lambda > 0\}$. The extremality of M^λ implies the set of effects $\{s_a^\lambda \mathrm{P}_a \mid a \in \mathscr{A}_\lambda\}$ to be linearly independent, and furthermore the condition $\sum_{a\in\mathscr{A}_\lambda} s_a^\lambda \mathrm{P}_a = \mathbf{I}$ uniquely specifies the values of s_a^λ's for any $\mathscr{A}_\lambda$. As the set $\{\mathrm{P}_a\}_{a=1}^k$ contains finitely many projectors, there are only finitely many ways of choosing $\mathscr{A}_\lambda$ such that $\mathbf{M}^\lambda$ turns out to be a rank-1 extremal POVM. Therefore, the integral in Eq.(D.23) gets replaced by finite summation, meaning

$$s_a \mathrm{P}_a = \sum_{\lambda=1}^{L} \mu_\lambda s_a^\lambda \mathrm{P}_a, \text{ with } \mu_\lambda > 0 \ \& \ \sum_{\lambda=1}^{L} \mu_\lambda = 1. \tag{D.24}$$

This completes the proof. □

Proof of Theorem [21]

Proof. Since any separable POVM is coarse-graining of rank-1 product POVMs, it suffices to prove our claim for the later only. Consider an K outcomes rank-1 product POVM

$$\mathbf{M} \equiv \left\{ p_i \mathrm{P}_{u_i} \otimes \mathrm{P}_{v_i} \mid |u_i\rangle \in \mathbb{C}_A^{d_1},\ |v_i\rangle \in \mathbb{C}_B^{d_2} \right\}_{i=1}^{K}. \tag{D.25}$$

Given a known state $|\psi\rangle \in \mathbb{C}^{d_1}$ to Alice and an unknown state $|\phi\rangle \in \mathbb{C}^{d_2}$ to Bob, they aim to reproduce the outcome statistics

$$p(i|\psi,\phi) := p_i \operatorname{Tr}[\mathrm{P}_{u_i}\mathrm{P}_\psi] \operatorname{Tr}[\mathrm{P}_{v_i}\mathrm{P}_\phi] \tag{D.26}$$

at Bob's laboratory. Denoting $p(i,\psi) = p_i \operatorname{Tr}[\mathrm{P}_{u_i}\mathrm{P}_\psi]$, the statistics in Eq.(D.26) can be view as the outcome statistics of the the effective rank-1 POVM $\mathbf{M}_{R1}^{\psi} := \{p(i,\psi)\mathrm{P}_{v_i}\}_{i=1}^{K}$ on Bob's unknown state $|\phi\rangle$. Lemma [5] ensures that POVM $\mathbf{M}_{R1}^{\psi} := \{p(i,\psi)\mathrm{P}_{v_i}\}_{i=1}^{K}$ can be expressed as probabilistic mixture of finite number of rank-1 extremal POVMs $\mathbf{M}^\lambda \equiv \{s_i^\lambda \mathrm{P}_{v_i}\}_{i=1}^{K}$, i.e.

$$p(i,\psi)\mathrm{P}_{v_i} = \sum_{\lambda=1}^{L} \mu_\lambda(\psi) s_i^\lambda \mathrm{P}_{v_i}. \tag{D.27}$$

More specifically Eq.(D.27) depicts that the coefficients $\{\mu_\lambda(\psi)\}_{\lambda=1}^{L}$ in convex mixture depend on the state of Alice's system. To simulate the statistics of Eq.(D.26) at Bob's end, Alice given a known state ψ generates a random variable $\lambda \in \{1,2,\cdots,L\}$ according to probability distribution $\{\mu_\lambda(\psi)\}_{\lambda=1}^{L}$ and communicates it to Bob using $\log L$-bits of classical communication. Upon receiving the random variable λ Bob implements the corresponding rank-1 extremal POVM $\mathbf{M}^\lambda$ on his unknown state ϕ. This completes the proof. □

D.7 Simulation of multipartite fully separable measurements

We start by introducing a natural multipartite generalization of the channel simulation task, that invokes more than one senders (say) Alice-1, Alice-2$\cdots$ Alice-n and one receiver Bob. Formally the task is defined as follows:

- Each of the senders is given classical description of qubit state, i.e., Alice-i receives classical description of the state $\psi_{A_i} \in \mathbb{C}^{d_i}_{A_i}$. Importantly, the knowledge of the state is known only to the i^{th} Alice, while it is oblivious to all other senders and the receiver.
- Bob receives an unknown quantum state $\phi_B \in \mathbb{C}^d_B$. Likewise the one sender-one receiver case, here also the state ϕ_B is unknown to the senders.
- Bob aims to reproduce statistics of a K outcome multipartite measurement $\mathrm{M}_{A_1\cdots A_nB} \equiv \left\{\Pi^b_{A_1\cdots A_nB}\right\}_{b=1}^K$ on the state $\bigotimes_{i=1}^n \psi_{A_i} \otimes \phi_B$, which reads as

$$p\left(b|\bigotimes_{i=1}^n \psi_{A_i} \otimes \phi_B\right) = \mathrm{Tr}\left[\left(\bigotimes_{i=1}^n \psi_{A_i} \otimes \phi_B\right)\Pi^b_{A_1\cdots A_nB}\right]. \tag{D.28}$$

Naturally, this raises the question of whether Theorem [21] can be generalized to fully separable measurements composed exclusively of fully separable effects [84]. Before proceeding further, it is important to note that in the multipartite setting, such measurements can exhibit nonlocal behavior in the sense that they may not be implementable within the operational paradigm of LOCC. A canonical example is the three-qubit Shift basis measurement [13]:

$$\mathrm{M}_{Shift} \equiv \left\{\begin{matrix} |000\rangle\langle 000|, & |111\rangle\langle 111|, & |{+}01\rangle\langle{+}01|,\ |{-}01\rangle\langle{-}01|, \\ |01{+}\rangle\langle 01{+}|, & |01{-}\rangle\langle 01{-}|, & |1{+}0\rangle\langle 1{+}0|,\ |1{-}0\rangle\langle 1{-}0| \end{matrix}\right\}, \tag{D.29}$$

where $|\pm\rangle := \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle$. Notably, Bennett et al. [13] demonstrated that the shift-basis measurement cannot be implemented via LOCC when all parties are spatially separated, though it becomes feasible if any two parties are in one laboratory. This construction was later extended to multipartite and higher-dimensional systems [169]. More recently, fully separable measurements have been identified that remain non-implementable under LOCC unless all parties are co-located, revealing a stronger form of measurement nonlocality [170–172].

This motivates an investigation into the applicability of Theorem [21] to such fully separable measurements.

While dealing with classical simulation of the statistics in Eq.(D.28), the following two configurations arise depending on how the classical resources are allowed:

- **Configuration A:** All the parties (the senders and the receiver) can exchange arbitrarily large but finite amount of classical communication among one another.
- **Configuration B:** Arbitrarily large but finite amount one-way classical communicate is allowed from each of the senders to the receiver. Back ward communication from the receiver to the senders as well as communication among the senders are not allowed.

Within the **Configuration A**, in the following we first show that it is possible to generalize Theorem [21].

Theorem 24. *Statistics of any fully separable measurement on a quantum states ψ_{A_i}, which is known to i^{th} Alice but unknown to others, and an unknown state provided to Bob, can always be simulated at Bob's end by finite classical communication allowed within Configuration A.*

Proof. The proof proceeds by induction. Assuming the result holds for $n-1$ senders, we show it must also hold for n. As in the bipartite case, it suffices to consider separable measurements performed by Bob, specifically those composed of rank-1 POVM elements. Let us consider a K-outcome, rank-1, fully product POVM of the form

$$\mathrm{M} \equiv \left\{ p_b \bigotimes_{i=1}^{n} \mathrm{P}_{u_b^i} \otimes \mathrm{P}_{v_b} \mid |u_b^i\rangle \in \mathbb{C}_{A_i}^{d_i},\ |v_b\rangle \in \mathbb{C}_B^{d} \right\}_{b=1}^{K}. \tag{D.30}$$

The outcome statistics of Eq.(D.28) reads as

$$\begin{aligned} p\left(b|\bigotimes_{i=1}^{n}\psi_{A_i}\otimes\phi_B\right) &= \mathrm{Tr}\left[\left(\bigotimes_{i=1}^{n}\psi_{A_i}\otimes\phi_B\right)\left(p_b\bigotimes_{i=1}^{n}\mathrm{P}_{u_b^i}\otimes\mathrm{P}_{v_b}\right)\right] \\ &= p_b\,\mathrm{Tr}\left[\psi_{A_1}\mathrm{P}_{u_b^1}\right]\times\mathrm{Tr}\left[\left(\bigotimes_{i=2}^{n}\psi_{A_i}\otimes\phi_B\right)\left(\bigotimes_{i=2}^{n}\mathrm{P}_{u_b^i}\otimes\mathrm{P}_{v_b}\right)\right]. \end{aligned} \tag{D.31}$$

Denoting $p(b,\psi_{A_1}) = p_b \operatorname{Tr}[\psi_{A_1} \mathrm{P}_{u_b^1}]$, the statistics in the above equation can be viewed as an effective rank-1 POVM separable measurement

$$\mathrm{M}_{R1}^{\psi_{A_1}} := \{p(b,\psi_{A_1}) \bigotimes_{i=2}^{n} \mathrm{P}_{u_b^i} \otimes \mathrm{P}_{v_b}\}_{b=1}^{K} \tag{D.32}$$

on the state $\bigotimes_{i=2}^{n} \psi_{A_i} \otimes \phi_B$. Lemma [5] ensures that POVM $\mathrm{M}_{R1}^{\psi_{A_1}}$ can be expressed as probabilistic mixture of finite number of rank-1 extremal POVMs $\mathrm{M}^{\lambda} \equiv \{s_b^{\lambda} \bigotimes_{i=2}^{n} \mathrm{P}_{u_b^i} \otimes \mathrm{P}_{v_b}\}_{b=1}^{K}$, i.e.

$$p(b,\psi_{A_1}) \bigotimes_{i=2}^{n} \mathrm{P}_{u_b^i} \otimes \mathrm{P}_{v_b} = \sum_{\lambda=1}^{L} \mu_{\lambda}(\psi_{A_1}) s_b^{\lambda} \bigotimes_{i=2}^{n} \mathrm{P}_{u_b^i} \otimes \mathrm{P}_{v_b}. \tag{D.33}$$

To classically simulate the required statistics, Alice-1 tosses a coin $\{\mu_{\lambda}(\psi_{A_1})\}_{\lambda=1}^{L}$ and communicates the outcome λ to the rest of the parties indicating that the rest of the parties should implement the measurement $\mathrm{M}^{\lambda} \equiv \{s_b^{\lambda} \bigotimes_{i=2}^{n} \mathrm{P}_{u_b^i} \otimes \mathrm{P}_{v_b}\}_{b=1}^{K}$ on their joint state $\bigotimes_{i=2}^{n} \psi_{A_i} \otimes \phi_B$. Thus for each λ we now have a similar separable measurement problem among the $n-1$ senders and one receiver. According to our inductive hypothesis every such measurement M^{λ} can be implemented by finite communication among the $n-1$ Alice's and Bob. Thus the measurement $\mathrm{M}_{R1}^{\psi_{A_1}}$ can also be simulated among the n senders and one receiver. The base case of the inductive proof for one sender and and one receiver follows from the Theorem [21]. This completes the proof. □

At this point, one might suspect that under restricted communication scenario (i.e., **Configuration B**) Theorem [24] may no longer hold. As we argue now this is not the case.

Theorem 25. *Statistics of any fully separable measurement on a quantum states ψ_{A_i}, which is known to i^{th} Alice but unknown to others, and an unknown state provided to Bob, can always be simulated at Bob's end by finite one-way classical communication from the senders to the receiver as allowed in Configuration B.*

Proof. We detail the argument for tripartite case, and the generalization follows for higher number of senders. Given two senders Alice-1 and Alice-2 and a receiver Bob let us consider a K-outcome, rank-1, fully product POVM of the form

$$\mathrm{M} \equiv \left\{ p_b \mathrm{P}_{u_b^1} \otimes \mathrm{P}_{u_b^2} \otimes \mathrm{P}_{v_b} \mid |u_b^1\rangle \in \mathbb{C}_{A_1}^{d_1}, |u_b^2\rangle \in \mathbb{C}_{A_2}^{d_2},\ |v_b\rangle \in \mathbb{C}_{B}^{d} \right\}_{b=1}^{K}. \tag{D.34}$$

As before, the effective POVM $\mathrm{M}_{R1}^{\psi_{A_1}} := \{p(b,\psi_{A_1})\mathrm{P}_{u_b^2} \otimes \mathrm{P}_{v_b}\}_{b=1}^{K}$ on the state $\psi_{A_2} \otimes \phi_B$ can be written as a probabilistic mixture of finite number of rank-1 extremal POVMs $\mathrm{M}^{\lambda} \equiv \{s_b^{\lambda}\mathrm{P}_{u_b^2} \otimes \mathrm{P}_{v_b}\}_{b=1}^{K}$.

In the case of **Configuration A**, Alice-1 communicates the information of λ to Alice-2 as well as Bob indicating them to simulate the statistics of M^{λ}. Whenever no communication between the senders is allowed, Alice-2 could just send the information corresponding to every possible value of $\lambda \in \{1 \cdots L\}$ since she has the complete classical description of the state ψ_{A_2}. Therefore, Alice-2 does not require the knowledge of the measurement M^{λ} to be simulated.

Therefore, the protocol in Theorem [24] can be modified by limiting Alice-1's communication of λ only to Bob.

Since Bob receives information for every possible value of λ from Alice-2, he can suitably choose the relevant information as indicated to him by Alice-1.

Following the aforementioned protocol, the tripartite statistics can be achieved in the case where the senders do not communicate between them. However, this comes at the cost of a large (but finite) amount communication from Alice-2 to Bob. It is not hard to see that the above argument generalizes for more than two senders. In this case, i^{th} Alice communicates all relevant information to Bob corresponding to Alice-$1, \cdots,$ Alice-$(i-1)$. □

D.8 Proof of Theorem [22]

Given the state $\psi := \frac{1}{2}(\mathbf{I}_2 + \hat{\psi} \cdot \sigma)$, the protocol ensures that the state $\mathbf{X}\mathrm{P}_{\hat{\omega}_{i^\star}}\mathbf{X}^{\dagger}$, prepared at Bob's end, lies within the cone forming an apex angle θ_m with the vector $\hat{\psi}$. Furthermore, since the random variable $\mathbf{X}$ is drawn Haar-randomly from the set of unitaries acting on $\mathbb{C}^2$, all the states within this cone are prepared with equal probability. Consequently, on average, Bob prepares a resulting density operator $\rho_R(\psi) = \frac{1}{2}(\mathbf{I}_2 + \vec{\psi}^R \cdot \sigma)$, with $\vec{\psi}^R = (\psi_x^R, \psi_y^R, \psi_z^R)^{\mathrm{T}} \in \mathbb{R}^3$.

Denoting the Bloch vector of $\mathbf{X}\mathrm{P}_{\hat{\omega}_{i^\star}}\mathbf{X}^\dagger$ as $(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)^{\mathrm{T}} \in \mathbb{R}^3$, the components of the resulting density operator are given by:

$$\psi_x^R = \frac{\int_0^{\theta_m} \sin\theta \, d\theta \int_0^{2\pi} d\varphi \, \sin\theta\cos\varphi}{\int_0^{\theta_m} \sin\theta \, d\theta \int_0^{2\pi} d\varphi}, \tag{D.35a}$$

$$\psi_y^R = \frac{\int_0^{\theta_m} \sin\theta \, d\theta \int_0^{2\pi} d\varphi \, \sin\theta\sin\varphi}{\int_0^{\theta_m} \sin\theta \, d\theta \int_0^{2\pi} d\varphi}, \tag{D.35b}$$

$$\psi_z^R = \frac{\int_0^{\theta_m} \sin\theta \, d\theta \int_0^{2\pi} d\varphi \, \cos\theta}{\int_0^{\theta_m} \sin\theta \, d\theta \int_0^{2\pi} d\varphi}. \tag{D.35c}$$

For instance, if Alice is given the state $|0\rangle\langle 0| = \frac{1}{2}(\mathbf{I}_2 + \sigma_z)$, then the components of the Bloch vector for Bob's resulting state are:

$$\psi_x^R = \psi_y^R = 0, \quad \text{and} \quad \psi_z^R = \frac{1}{4} \cdot \frac{1-\cos 2\theta_m}{1-\cos\theta_m} := \eta(\theta_m). \tag{D.36}$$

Thus, the resulting density operator at Bob's end is:

$$\rho_R(|0\rangle) = \frac{1}{2}\left(\mathbf{I}_2 + \eta(\theta_m)\,\sigma_z\right). \tag{D.37}$$

The above calculation yields the same result for any arbitrary state ψ provided to Alice, confirming that the protocol simulates a depolarizing channel with parameter $\eta(\theta_m)$. Notably, as m increases, θ_m decreases, and $\eta(\theta_m)$ increases. In the limiting case $m \to \infty$, we have $\theta_m \to 0$ and $\eta(\theta_m) \to 1$, which aligns with the claims of Theorem [18]. Thus, for any value of $\eta < 1$, a simulation is always achievable with m bits of finite communication, provided m is sufficiently large. In general, deriving an exact expression for $\eta(\theta_m)$ for arbitrary m is challenging, as it depends on the specific choices of Bloch vectors $\{\hat{\omega}_i\}_{i=1}^{2^m}$. However, for small m's we can have some natural choices of Bloch vectors – (m=1): 2 diametrically opposite vectors, yielding $\theta_1 = \pi/2$ and $\eta(\theta_1) = 1/2$, (m=2): 4 vectors forming a regular tetrahedron, yielding $\theta_2 = 0.5 \times \cos^{-1}(-1/3)$ and $\eta(\theta_2) = (3+\sqrt{3})/6 \approx 0.789$, and ($m$=3): 8 vectors forming the vertices of a cube, yielding $\theta_3 = 0.5 \times \cos^{-1}(1/3)$ and $\eta(\theta_3) = (3+\sqrt{6})/6 \approx 0.908$.